Dynamic Risk Management of Commodity Operations: Model and Analysis

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Dynamic Risk Management of Commodity Operations: Model and Analysis

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We consider the dynamic risk management problem for a commodity processor facing risk costs. The firm procures an input commodity and processes it to produce an output commodity over a multi-period horizon. The processed commodity is sold using forward contracts while the input itself can be traded at the end of the horizon. The firm can also trade financial instruments to manage the commodity price risk, but cannot hedge the risk completely. Using the concept of conditional risk mappings, we extend the single period conditional value at risk (CVaR) measure to a dynamic setting and ensure a time-consistent representation of the firm’s risk management objective. In a partially complete market framework, we show that the optimal financial trading policy is a CVaR–replicating portfolio. Contingent on optimal financial trading, we show that the optimal procurement and processing policies are characterized by price and horizon dependent inventory thresholds. We show, analytically, that the procurement threshold increases with risk costs during the horizon. Our numerical studies show that optimizing a time-consistent risk measure results in better risk control over the entire horizon when compared to optimizing the CVaR of total profits. We also find that myopic and risk-neutral optimal policies are poor substitutes for the optimal risk management policy, especially when the firm faces significant risk costs.

Key words: Risk management, storable commodities, time-consistent risk measures, partially complete markets

1. Introduction

In this paper, we consider the risk management problem for a commodity processing firm affected by commodity price uncertainty. Firms face many uncertainties, including price uncertainty, that make profits/cashflows in future periods uncertain. In states where cashflows are low, firms may incur risk costs due to financial distress. These risk costs include actual costs and/or missed opportunities. For example, a financially distressed firm might lose customers, important suppliers and employees, incur heavy financial penalties because of missed debt repayments, or may have to forego investment opportunities due to costly external financing (Purnanandam 2008). As a consequence, firms take active measures to deal with uncertainties and mitigate risk costs. With the growth of commodity exchanges, a firm can combine operational decisions with trading in the financial markets to better manage uncertainty. Because changes in cashflow patterns affect risk
costs, it is important to understand how combining financial trading with operational decisions affects overall cashflow patterns and what impact such trading has on the operational policies.

Markets are said to be complete, when any random cashflow whose realization is contingent on the future state of the world can be replicated exactly by trading a portfolio of market securities (Smith 1998). As a result, any risk costs that the firm incurs can also be replicated and, therefore, hedged away by trading financial instruments. When combined with the optimal financial trading policy, the value of cashflows generated by any operating policy is equal to the present value of a portfolio that dynamically replicates those cashflows. Thus, irrespective of whether the risk costs faced by the firm are significant or not (cases (SC) and (NC) in Table 1), the firm’s optimal operational policy should maximize the market value of the replicating financial portfolio. Under standard assumptions, the value of the replicating portfolio is the expected value of the random cashflows discounted at the risk-free rate, where the expectation is taken under a risk-adjusted probability measure called the risk-neutral probability measure (cf. Smith and Nau 1995, Smith 1998).

More often than not, markets are incomplete; i.e., it is not possible to replicate all random cashflows in every possible future state of the world using market securities. For instance, commodity processing firms may face uncertain demand and market securities that replicate cashflows for different demand realizations may not be available. Even if the firm does not face uncertain demand, very often commodity spot prices are affected by what is known as basis risk and the actual spot price is not perfectly correlated with the price of market securities. Differences between the local spot and market price may exist because of firm specific or private uncertainties, such as timing, location and quality discrepancies (cf. Paroush and Wolf 1989, Moschini and Lapan 1995).

If the risk costs faced by the firm are not significant, the approach of maximizing expected value of random cashflows under a risk-adjusted probability measure is valid even when the markets are incomplete (case (NI) in Table 1). By trading market securities, the firm can replicate the expected value of the random cashflows generated by any operating policy, where the expectation is taken over those sources of uncertainties that cannot be hedged. As in the complete markets case, the firm’s optimal operational policy should then maximize the present value of this replicating portfolio. Depending on the source of incompleteness and its degree of correlation with the market, a firm can determine an appropriate risk-adjusted probability measure and the value of the replicating portfolio is the expected value of the random cashflows under this risk-adjusted measure (cf. Smith and Nau 1995, Smith 1998, for details).
When markets are incomplete and the firm faces significant risk costs, it is important to account for the risk costs in the firm’s objective function and determine optimal financial and operational policies for risk management (case (SI) in Table 1). While there is sufficient literature that supports the need for risk management by firms to manage risk costs (see Smith and Stulz 1985, Froot et al. 1993, Bickel 2006, for instance), modeling these risk costs themselves is very difficult. For instance, how costs of borrowing increase as a function of capital shortfall, or how financial distress costs faced by a firm vary as a function of the firm’s cashflows are very difficult to model and calibrate (Brown and Toft 2002). In practice, firms use shortcuts to approximate these costs using different functions of the firm’s profits/cashflows. An approach that is widely used in the commercial banking industry, and now in many firms even outside the financial industry, is to approximate these risk costs by risk measures such as value-at-risk (VaR) or conditional value-at-risk (CVaR). VaR is the level of losses that is exceeded only with a small probability, say 1%, or alternately the value below which profits fall only with a small probability (see Duffie and Pan 1997, for a detailed discussion on VaR). CVaR is an enhancement of VaR and is roughly equal to the expected conditional loss when the loss is greater than or equal to the VaR, or the expected profits when profits are less than or equal to the VaR (see Rockafellar and Uryasev 2000, for a precise definition). The impact of risk management objectives is further complicated by the fact that firms usually wish to devise policies for multi-period horizons. Risk measures on the net present value (NPV) of total wealth are often used to model the risk management objective over multiple periods. Such an approach, however, is not always effective in accounting for the impact of uncertainty resolution in intermediate periods, and can often result in time-inconsistent operational policies; viz., policies that were deemed optimal even one period earlier, for a state that had been fully accounted for when the decision was made, may be reversed when the actual state is realized (a brief overview of risk measures and time consistency issues in multi-period risk measures is provided in Appendix B). Thus, a practical consideration for a firm operating in incomplete markets and facing risk costs is to represent its risk management objective by a time-consistent risk measure and devise appropriate operational and financial trading policies, which is the focus of this paper.

We consider a commodity processing firm that operates over multiple periods and is interested in devising an operational (i.e., procurement, processing and output commitment) policy, along with a financial trading policy, to manage risk over the entire horizon. For the risk management problem to be relevant, we consider a situation where the firm incurs risk costs in some states of
the world and faces commodity price risk that cannot be completely hedged using market instruments. Many commodity processing and trading firms, among others, face significant risk costs. For instance, VeraSun Energy (VSE), a bio-fuel company and one of the biggest producers of ethanol, was forced to file for bankruptcy because it found itself on the wrong side of both the corn and the ethanol markets (Mandaro 2008). Further, these firms often operate in incomplete markets. For instance, actual spot prices of commodities in the commodity producing country may not be perfectly correlated with market prices of financial instruments (Dana and Gilbert 2008).

To model the firm’s risk management objective function for the planning horizon, we use the concept of conditional risk maps (Ruszczynski and Shapiro 2006) to develop a time-consistent risk measure based on the single period CVaR measure. In each period, contingent on the realized commodity prices, the firm decides how much (commodity) input to procure, and how much of the available input to process, subject to capacity constraints. The firm uses forward contracts to sell the output and chooses the quantity to commit for sale in each period. We model market incompleteness by considering a partially complete market (Smith and Nau 1995), where the input commodity price is a function of both market prices and firm specific factors. We find that for each market state, the optimal financial trading policy replicates the CVaR, instead of expected value, of operational cashflows over the states of private uncertainty. The firm’s optimal operational policy maximizes the present market value of this replicating portfolio. We show that it is optimal to postpone all output sale commitments against any forward contract to the last possible period and the optimal procurement and processing policies are governed by price and horizon dependent thresholds. Somewhat surprisingly, we find that an increase in risk costs in intermediate periods leads to higher input procurement thresholds. To the best of our knowledge, this is one of the very few, if not the only, comparative statics results of operational decisions with respect to risk management in a multi-period setting.

We supplement our analytical results with extensive numerical studies. We show that using a policy that maximizes a time-consistent objective function leads to a better risk profile of minimum

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<td>(NC): Maximize value of portfolio replicating cashflows</td>
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<td>Significant (S)</td>
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wealth over the horizon, compared to a policy that maximizes a static measure on the NPV of total wealth. Further, the improvement in the risk profile of minimum wealth is not accompanied by any significant loss in the expected value of terminal wealth. Our numerical study also shows that myopic policies and policies that are optimal for risk-neutral firms are poor substitutes for the optimal risk management policy. Fortunately, computing the optimal operational policy is not significantly more burdensome than computing the optimal risk-neutral policy.

The remainder of the paper is organized as follows. In the next section, we discuss relevant literature related to our work. In Section 3, we describe our partially complete market framework and define a time-consistent risk measure. Section 4 models the operations of a commodity processing firm. Optimal operational policies are described in Section 5. Section 6 provides numerical illustrations to quantify the benefit of using a time-consistent objective function, impact of risk management considerations on performance, and show comparative assessment of heuristics for a firm facing risk costs. We conclude in Section 7 with directions for future research. A summary of the most frequently used notation is given in Appendix A. A brief overview of risk measures and time consistency of multi-period risk measures is provided in Appendix B, while proofs for the analytical results are given in Appendix C.

2. Literature Review

In recent years, there has been considerable research related to commodity operations in the Operations Management (OM) area. These papers, however, mostly deal with a situation where the markets are complete and/or the firm does not face risk costs (cases (NC), (SC) and (NI) in Table 1). Secomandi (2010) and Lai et al. (2010) for instance, are papers that consider complete markets and determine the optimal operational policy for a natural gas storage facility. Goel and Gutierrez (2006) and Berling and Martínez-de Albéniz (2011) are papers where the market is incomplete, but the firm faces no significant risk costs. They use the approach of maximizing expected value of operational cashflows under a risk-adjusted probability measure for firms using commodities as inputs in their process. Both these papers consider the optimal input procurement and inventory management policies for a firm facing stochastic demand and fluctuating commodity prices. Devalkar et al. (2011) is closely related to our work but falls in the (NC) category, where the authors consider operations of a risk-neutral firm in the presence of uncertain commodity prices and procurement and processing constraints. The firm procures an input commodity, processes the input to produce output commodities that are then traded.
The OM literature related to risk management (category (SI) in Table 1) is fairly sparse. Early work took an utility function approach to model risk-averse preferences with expected utility as the criterion for decision making in these contexts; (for instance, see Eeckhoudt et al. 1995, Agrawal and Seshadri 2000, who use utility functions to model risk-averse newsvendor problems). These papers focus exclusively on operational decisions and treat a firm as an entity or an individual that maximizes an expected utility function. A firm’s risk preferences, however, are rarely defined in terms of a corporate utility function. Smith (2004) notes that in a normative sense, there is no ‘right’ corporate utility function that can be used, and suggests approaches to model this; e.g., using individual unit managers’ preferences or shareholders’ utility function. An additional factor, as discussed in Brown and Toft (2002), is that the hedging decisions that are optimal for a firm that wishes to hedge purely to avoid risk costs can be very different from the optimal decisions for a risk-averse utility maximizer.

More recent papers in the (SI) category use coherent risk measures to model risk-averse decision making / risk management objectives. Most of these papers, however, consider single period settings; see for example, Gotoh and Takano (2007) and Gao et al. (2011) for newsvendor problems, and Tomlin (2006) in the context of flexibility and supply disruptions. Further, these papers do not consider financial hedging decisions. Research that integrates operational and financial decision making has also typically dealt with single period problems. Kleindorfer (2008) provides a good review of the literature on single period models that integrate commodity markets and operational decisions to manage risk. Where multi-period problems are considered, the literature deals primarily with non-storable commodities, where the operational problem can be decomposed into a sequence of single period problems where time consistency issues are not relevant, thus simplifying the analysis (Kleindorfer and Li 2005, Zhu and Kapuscinski 2006). In contrast, we model a firm’s risk management problem in a multi-period setting where operational decisions in one period affect cashflows and decisions in future periods and therefore explicitly consider the impact of using time-consistent risk measures on the firm’s optimal operational policy.

Like us, Geman and Ohana (2008) and Kouvelis et al. (2011) consider a firm that is involved in commodity operations and manages risk over a multi-period horizon. While these two papers, and ours, fall under category (SI) in Table 1, there are however key differences. Geman and Ohana (2008) focus on formalizing the properties of time-consistent measures and restrict themselves to illustrate the benefits of using time-consistent measures. In contrast, we provide detailed analytical
results for the optimal financial and operational policies that are useful for firms in making day-to-day decisions to manage their procurement and processing operations. A significant difference between our paper and Kouvelis et al. (2011) is in terms of how the firm’s objective function is modeled. Kouvelis et al. (2011) use a mean-variance objective function on the NPV of profits. To ensure time consistency, their formulation requires a large state space that includes the current inventory, wealth, commodity prices, and the decisions in every possible state of the world in all the future periods. In contrast, we use a time-consistent risk measure that allows us to model the firm’s decision problem at any stage using a Bellman-type recursive equation that requires a state space consisting of just the current inventory level and commodity prices. Further, our numerical studies show that using a time-consistent risk measure leads to better performance in terms of the risk profile of cashflows obtained over the entire horizon, compared to optimizing a time-inconsistent measure. Similar to our approach, Ahmed et al. (2007) construct a dynamic time-consistent risk measure using conditional risk maps (Ruszczynski and Shapiro 2006) to model a multi-period risk-averse stochastic inventory management problem. However, the context in Ahmed et al. (2007) is different, where uncertainty in cashflows is due to stochastic demand. Further, they do not consider joint operational and financial decisions like we do in the current paper.

3. Time-consistent Risk Measures and Financial Hedging in Partially Complete Markets

In this section, we describe the partially complete market setup we use to model uncertainty, discuss the role of risk management / hedging in incomplete markets and introduce a time-consistent risk measure to model a firm’s objective function. We also derive a key result regarding financial hedging that will prove useful in our analysis of the commodity processing firm’s operations.

3.1. Partially Complete Markets

We use the framework of partially complete market to model uncertainty: the uncertainty in a future state of world consists of market related uncertainty and private, or firm specific, uncertainty. For instance, the commodity spot price could be a function of the price of financial instruments linked to the commodity and local factors such as quality, timing and location. Any payoff that depends only on market related uncertainty can be hedged perfectly in the financial markets, whereas those payoffs that depend only on private uncertainty cannot be hedged using financial instruments. Formally, we assume the period $n$ state of information $I_n$ can be written as a vector of market and private states of information $I_n = (I_{nm}^n, I_{np}^n)$, specified by discrete and finite probability spaces. The market is partially complete if the following conditions are satisfied (Smith and Nau 1995):
1. Security prices depend only on the market states and can be written as a function of the market state of uncertainty.

2. The market is complete with respect to market uncertainties; i.e., the security prices span the space of cashflows dependent only on the market states.

3. Private events convey no information about future market events; i.e., given \( I_{m-1} \), the firm believes that \( I_{m} \) and \( I_{p_{n-1}} \) are independent.

Let \( \bar{M}_n(I_m) = (M^0_n(I_m), \ldots, M^J_n(I_m)) \geq \bar{0} \) be the vector of the period \( n \) prices of the \( J + 1 \) market securities and \( \beta = \frac{1}{1+r_f} \) the risk-free discount rate. Conditions 1 and 2 above imply that there exist unique risk-neutral probabilities \( \pi_n(I_{m+1} | I_m) \) on the market states \( I_{m+1} \), conditional on \( I_m \), such that

\[
\beta \sum_{I_{m+1} \in I_m} \pi_n(I_{m+1} | I_m) \bar{M}_{m+1}(I_{m+1}) = \bar{M}_m(I_m).
\]

Aside from the risk-neutral probabilities implied by the prices of the market securities, the firm may have its own subjective probabilities on the market states of uncertainty which are different from the risk-neutral probabilities. We denote these subjective probabilities by \( \gamma_n(I_{m+1} | I_m) \). Also, the firm’s subjective probabilities over the private states of uncertainty in period \( n \), conditional on the market state \( I_m \), are denoted by \( q_n(I_{n+1} | I_m) \) (To keep the notation manageable, we will use the symbols defined above without explicitly showing their arguments except when necessary; e.g., we will denote \( \pi_n(I_{m+1} | I_m) \) as simply \( \pi_n \), in the remainder of the paper).

Many models of commodity spot price express it as a function of the settlement price of market instruments and other factors, which are assumed to be stochastically separable or independent (see Mahul 2002, Moschini and Lapan 1995, for example). These models of commodity spot prices are well represented by the partially complete market framework described above.

### 3.2. Dynamic Risk Measure

Modeling risk costs is usually very difficult. As a result, these costs are often approximated as functions of the firm’s profits or cashflows. For instance, Brown and Toft (2002) model deadweight costs associated with financial distress as an exponential function of the firm’s profits, while Pur- nanandam (2008) models the cost of financial distress as a permanent reduction in its final value. Using risk measures on the firm’s profits (or cashflows) to model the costs imposed by uncertainty in profits generalizes this approach. While Appendix B provides a brief overview of coherent risk measures and how they are useful to model risk management objective functions, we focus specifically on the CVaR risk measure and its dynamic time-consistent counterpart, labeled DCVaR, to derive insights about the risk management problem. CVaR at a given probability level \( \eta \in (0, 1] \)
represents the expected value in the worst $\eta$ proportion of the cases. For a random variable with continuous distribution, CVaR at the $\eta$ level represents the expected value in the left $\eta$ tail of the distribution. A higher value of $\eta$ denotes lower risk costs and when $\eta = 1$, CVaR is equal to the expected value.

To understand how CVaR is relevant in an operational context, consider the example of a newsvendor who makes her stocking decision before demand is known. The final cashflow realized by the firm, $X(\xi, Q)$, is a random variable whose distribution depends on the distribution of the demand $\xi$ as well as the stocking quantity $Q$. In a risk-neutral setting, the newsvendor chooses $Q$ to maximize the expected value of $X(\xi, Q)$. That is, maximize CVaR at $\eta = 1$. Similarly, when the newsvendor faces risk costs she can choose $Q$ to maximize CVaR of $X(\xi, Q)$ at the appropriate $\eta < 1$ level.

Formally, let $X(I_1)$ be a random variable that is measurable with respect to information $I_1$ that is revealed at the end of the period and $I_0$ denote the information available at the beginning of the period. Then, the CVaR of $X(\cdot)$ at the $\eta$ level for $\eta \in (0, 1]$, denoted by $CVaR_\eta(X(I_1)|I_0)$, can be written as follows (Rockafellar and Uryasev 2000)

$$CVaR_\eta(X(I_1)|I_0) = \max_\nu \left\{ \nu - \frac{1}{\eta} \mathbb{E} \left[ (\nu - X(I_1))^+ | I_0 \right] \right\}$$

(1)

where $\mathbb{E}[\cdot]$ denotes expectation over $I_1$ conditional on $I_0$, using the firm’s subjective probability measure (the expression for CVaR above considers $X(\cdot)$ to denote profits or gains, while the definition in Rockafellar and Uryasev (2000) is based on $X(\cdot)$ denoting losses).

Extending the newsvendor example to a multi-period setting, we can think of a stream of cashflows $\vec{X} = (X_1, \ldots, X_N)$, where $X_n$ is the random cashflow in period $n$ that depends on $I_n$, the demands realized and stocking quantity decisions from periods 1 to $n$. One way to model the multi-period risk management problem for a firm facing risk costs would be to choose a policy, i.e., stocking quantities $Q_1(I_0), \ldots, Q_N(I_{N-1})$, to maximize $CVaR_\eta \left( \sum_{\tau=1}^N X_\tau | I_0 \right)$. As discussed in the introduction and previous literature (see Geman and Ohana 2008), such an objective function may not be time-consistent (see Appendix B for a discussion about time consistency). Using the concept of conditional risk maps (Ruszczynski and Shapiro 2006), we propose a natural, time-consistent counterpart of $CVaR_\eta \left( \sum_{\tau=1}^N X_\tau | I_{n-1} \right)$, labeled DCVaR, as an objective function to help the firm manage risk costs over multiple periods.

In our model of a multi-period partially complete market, for each period $n$ let $\mathcal{X}_n$ and $\mathcal{X}_n^m$ denote the set of all $I_n$- and $I_n^m$-measurable functions respectively. Let $CVaR^{(p)}_{(\eta,n)} : \mathcal{X}_n \rightarrow \mathcal{X}_n^m$ be
such that $CVaR_{(\eta,n)}(X_n) \triangleq CVaR_\eta(X_n|\mathcal{I}_n^m)$ for all $X_n \in \mathcal{X}_n$; i.e., $CVaR_{(\eta,n)}(X_n)$ is a scalar conditional risk map which evaluates the expected value of $X_n$ in the worst $\eta$ proportion of cases because of private uncertainty, conditional on the realized market uncertainty $\mathcal{I}_n^m$. For instance, if $X_n \in \mathcal{X}_n$ is the cashflow in period $n$ that depends on both the private and market uncertainties, then $CVaR_{(\eta,n)}(X_n)$ is a measure of the riskiness in $X_n$ because of the private uncertainty.

Similarly, let $CVaR_{(\eta,n)}^{(m)} : \mathcal{X}_{n+1}^m \rightarrow \mathcal{X}_n^m$ be a risk map such that $CVaR_{(\eta,n)}^{(m)}(X_{n+1}^m) \triangleq CVaR_\eta(X_{n+1}^m|\mathcal{I}_n^m)$ for all $X_{n+1}^m \in \mathcal{X}_{n+1}^m$; i.e., $CVaR_{(\eta,n)}^{(m)}(X_{n+1}^m)$ is a scalar conditional risk map that evaluates the expected value of $X_{n+1}^m$ in the worst $\eta$ proportion of cases because of market uncertainty, conditional on the realized market uncertainty $\mathcal{I}_n^m$. That is, for a cashflow that depends only on market uncertainty, $CVaR_{(\eta,n)}^{(m)}(X_{n+1}^m)$ is a measure of the riskiness because of next period’s market uncertainty.

To understand how the above defined scalar conditional risk maps can be used to define a dynamic risk measure, consider an example where a firm needs to sell a certain commodity in a local market in period $N$. The spot price, $S_N$, at which the commodity can be sold is affected by market and private uncertainties. Suppose the firm has access to market information, i.e., knows the realization of $\mathcal{I}_N^m$, but not the state of private uncertainty. Conditional on $\mathcal{I}_N^m$, the expected value of the price at which it can sell the commodity in the worst $\eta$ proportion of cases would then be equal to $CVaR_{(\eta,N)}(S_N)$. Now, suppose the firm needs to acquire this commodity in period $N-1$. At the time of acquiring this commodity, the firm has access to market information of period $N-1$, i.e., knows the realization of $\mathcal{I}_{N-1}^m$. In order to evaluate the expected value at which it can sell the commodity in the worst $\eta$ proportion of cases, it needs to account for both market and private uncertainties in period $N$. Now, given a realization of $\mathcal{I}_N^m$, we know the worst case expected price is $CVaR_{(\eta,N)}(S_N)$. However, $CVaR_{(\eta,N)}(S_N)$ is a function of $\mathcal{I}_N^m$ and is uncertain. Thus, the firm needs to account for the market uncertainty when computing the expected price in the worst $\eta$ proportion of cases in the next period. Furthermore, to be consistent with how private uncertainty is conditional on the realized market uncertainty in period $N$, the firm should evaluate $CVaR_{(\eta,N)}(S_N)$ in the worst $\eta$ proportion of cases of $\mathcal{I}_N^m$, given $\mathcal{I}_{N-1}^m$. That is, similar to taking iterated expectations, the firm can calculate $CVaR_{(\eta,N)}^{(m)} \left( CVaR_{(\eta,N)}(S_N) \right)$ to quantify the expected price at which it can sell the commodity in the worst $\eta$ proportion of cases in period $N$, conditional on $\mathcal{I}_{N-1}^m$.

More generally for a stream of cashflows $\tilde{X} = (X_1(\mathcal{I}_1), \ldots, X_N(\mathcal{I}_N))$, where $X_n(\cdot)$ is the cashflow in period $n$, we can use the above defined scalar conditional risk maps, to define a dynamic risk measure $DCVaR_{(\eta,n)}$, for $n = 1, \ldots, N$, as follows.
DCVaR(\(\vec{\eta}, N\)) \(\left(\vec{X}, \mathcal{T}^m_N\right) = CVaR_{(\eta, n)}^{(p)}(X_n) \) (2)

DCVaR(\(\vec{\eta}, n\)) \(\left(\vec{X}, \mathcal{T}^m_n\right) = CVaR_{(\eta, m)}^{(p)}(X_n) + CVaR_{(\eta, n)}^{(m)} \left(\beta DCVaR_{(\vec{\eta}, n+1)}(\vec{X}, \mathcal{T}^m_{n+1})\right) \) (3)

where \(\vec{\eta} = (\eta_1, \ldots, \eta_N)\) is the vector of probability levels such that \(\eta_n \in (0, 1]\) is the probability level at which \(CVaR_{(\eta, n)}^{(p)}(\cdot)\) and \(CVaR_{(\eta, n)}^{(m)}(\cdot)\) are evaluated, for each \(n\).

From equation (3), we can see that DCVaR(\(\vec{\eta}, n\))(\(\vec{X}, \mathcal{T}^m_n\)) is a measure of the riskiness of the cumulative cashflow \(X_n(\cdot) + \ldots + X_N(\cdot)\). In that sense, it is similar to a measure such as \(CVaR_{\vec{\eta}} \left(\sum_{\tau=n}^{N} X_{\tau}(\cdot)\right)\). By evaluating the riskiness at each stage conditional on the realized uncertainty, DCVaR(\(\vec{\eta}, n\)) ensures a time-consistent evaluation of risk, whereas \(CVaR_{\vec{\eta}}(\cdot)\) will not necessarily lead to a time-consistent evaluation.

### 3.3. Hedging using Market Securities

Suppose the cash flow stream \(\vec{X}\) described in the previous section is generated by the firm’s operations. By trading market securities, the firm can hedge the market driven uncertainty in the cashflow stream and potentially reduce the risk costs it incurs. Let \(\vec{\alpha}_n = (\alpha_n^0, \ldots, \alpha_n^J)\) be the firm’s position in the various market securities after observing the state of market uncertainty in period \(n\), with \(\vec{\alpha}_0 = (0, \ldots, 0)\). Let \(B = (\vec{\alpha}_1, \ldots, \vec{\alpha}_{N-1}, \vec{\alpha}_N)\) be the trading policy for market securities that is adapted to the filtration generated by \(\mathcal{T}^m_n\), and \(\vec{Y}^m = ([\vec{\alpha}_1]^T \vec{M}_1, [\vec{\alpha}_1 - \vec{\alpha}_2]^T \vec{M}_2, \ldots, [\vec{\alpha}_{N-1} - \vec{\alpha}_N]^T \vec{M}_N, [\vec{\alpha}_N]^T \vec{M}_{N+1})\) be the cashflows generated by the financial trading policy (although there are no operational cashflows in period \(N + 1\), there are cashflows because of financial trading).

We use DCVaR (equations (2)–(3)) to represent the firm’s risk management objective function. CVaR is a coherent (therefore monotonic) risk measure, so the firm’s optimal financial trading policy for a given \(\vec{\eta}\) can be determined by solving the following stochastic dynamic program (SDP).

\[
\mathcal{H}_n(\vec{X}, \vec{\alpha}_{n-1}, \mathcal{T}^m_n) = \max_{\vec{\alpha}_n} \left\{ \left[\vec{\alpha}_{n-1} - \vec{\alpha}_n\right]^T \vec{M}_n + CVaR_{(\eta, n)}^{(p)}(X_n) \right. \\
\left. + CVaR_{(\eta, n)}^{(m)}(\beta \mathcal{H}_{n+1}(\vec{X}, \vec{\alpha}_n, \mathcal{T}^m_{n+1})) \right\} \tag{4}
\]

\[
\mathcal{H}_N(\vec{X}, \vec{\alpha}_{N-1}, \mathcal{T}^m_N) = \max_{\vec{\alpha}_N} \left\{ \left[\vec{\alpha}_{N-1} - \vec{\alpha}_N\right]^T \vec{M}_N + CVaR_{(\eta, N)}^{(p)}(X_N) \right. \\
\left. + CVaR_{(\eta, N)}^{(m)}(\beta [\vec{\alpha}_N]^T \vec{M}_{N+1}) \right\} \tag{5}
\]

Under some mild restrictions on \(\vec{\eta}\) and the firm’s subjective probabilities, the value function \(\mathcal{H}_n(\cdot)\) of the above maximization problem has a particularly simple form, given in Theorem 1 below. This parallels the result obtained by Smith and Nau (1995) who, in addition to the partially
complete market structure, impose the additional restriction that a firm’s risk aversion is represented by an additive exponential utility function over net cashflows.

**Theorem 1.** Suppose the firm has no budget constraints and the firm’s subjective probabilities \( \gamma_n \) over market uncertainty and \( \eta_n \) are such that \( \frac{\gamma_n}{\eta_n} \geq \pi_n \) for each \( (I_{m+1,n}, I_{m,n}) \), for all \( n \). Then, the optimal financial trading policy for the firm is a \( \text{CVaR}(p) \) \( (\eta_n,n) \)-replicating portfolio for all \( n \), for all \( I_{m,n} \). Furthermore,

\[
H_n(\vec{X}, \vec{\alpha}_{n-1}, T_n^m) = (\vec{\alpha}_{n-1})^T \vec{M}_n + \mathbb{E}_{\pi,n} \left[ \sum_{\tau=n}^{N} \beta^{\tau-n} \text{CVaR}(p)(\eta_{\tau,n}, \tau) (X_{\tau}) \right]
\]

where \( \mathbb{E}_{\pi}[\cdot] \) denotes expectation over market states under the risk-neutral probability measure.

Theorem 1 implies that for each \( n < N \), for each \( I_{m,n} \), the optimal positions in the market securities, \( \vec{\alpha}_n^* \), are such that \( \text{CVaR}(p)(\eta_n,n) (X_n + [\vec{\alpha}_n^* - \vec{\alpha}_{n-1}]^T \vec{M}_n) \) is the same for all \( I_{m,n} \). In other words, while optimal financial trading does not eliminate all uncertainty in cashflows, it ensures that the worst case expected performance over private states is the same across different market states in a given period.

To understand the restriction on the firm’s subjective probabilities and risk-neutral probabilities, consider a firm that does not face any risk costs, i.e., \( \eta_n = 1 \) for all \( n \). The condition in Theorem 1 reduces to \( \gamma_n = \pi_n \) and we can see from the proof of Theorem 1 that the firm will be able to create a portfolio with a strictly positive expected payoff if \( \gamma_n \neq \pi_n \). As a result, the firm can earn unbounded profits in expectation by simply trading in the financial markets. In the more general case where the firm faces risk costs and its objective is represented by the DCVaR measure, the restriction on the firm’s subjective and risk-neutral probabilities on market states of uncertainty is a form of no-arbitrage condition to rule out any speculative motive for trading in the financial markets. This condition is consistent with empirical research that shows there is no evidence of systematic speculation by firms (Hentschel and Kothari 2001). It is also a common assumption in the extant literature related to financial hedging by firms (Brown and Toft 2002).

Now, consider a situation where the project cashflows \( \vec{X} \) depend on an operational policy and the firm is interested in optimizing over the joint operational and financial policy. Theorem 1 implies that this joint optimization can be carried out in a sequential manner: the optimal operational policy can be determined first as the policy that maximizes

\[
\mathbb{E}_{\pi_1} \left[ \sum_{n=1}^{N} \beta^{n-1} \text{CVaR}(p)(\eta_{n,n}) (X_n) \right],
\]

and subsequently, the optimal financial trading policy is the \( \text{CVaR}(p)(\eta_{n,n}) \)-replicating portfolio of the operational cashflows generated by the optimal operational policy.
Before we describe the operations of a commodity processing firm and apply the above results, we wish to re-iterate the fact that we use DCVaR as the dynamic risk measure for expositional clarity. All the analysis, and previous (and subsequent) structural results will continue to hold if we define a time-consistent risk measure along the lines of equations (2)–(3), where the basic building block is any general, single period coherent risk measure and the no-arbitrage condition on subjective probabilities in Theorem 1 is appropriately modified.

4. Model Description and Analysis

We consider a firm that procures, processes and trades commodities over a finite horizon. For instance, consider the operations of agricultural commodity processors where the input commodity has a finite well identified growing season over which the input is procured and processed. In each period \( n \), the firm procures the input commodity from a spot market at price \( S_n(I_n) \). The firm earns revenues by processing the input and selling the output commodity (processed product) using forward contracts of different maturities. The forward price on contract that matures in period \( N_\ell \) is denoted by \( F_n^{N_\ell}(I_{m}^n) \), where \( N_\ell > n \). The delivery period for all commitments made against contract \( N_\ell \) is \( N_\ell \) and \( N_\ell - 1 \) is the last period in which the firm can commit to sell the output commodity using the forward contract \( N_\ell \). We use \( \vec{F}_n(I_{m}^n) = (F_n^{N_1}(I_{m}^n), \ldots, F_n^{N_L}(I_{m}^n)) \) to denote the vector of forward prices of all contracts yet to mature as of period \( n \) (in the following, we suppress the dependence of \( F_n^{N_\ell} \) and \( \vec{F}_n \) on \( I_{m}^n \), and of \( S_n \) on \( I_n \), to keep the notation manageable). We assume forward prices are pegged to the price of actively traded futures instruments on the output commodity and therefore, the forward prices depend only on the state of market uncertainty. We assume the firm only produces output of the quality required by the underlying futures instruments.

In addition to the output commodity sales, the firm can also earn revenues by trading the input commodity with other processors. For ease of exposition, we assume that all input commodity trading occurs at the end of the horizon, at the trade (salvage) price of \( S_N(I_N) \). (Including opportunities to trade the input commodity in intermediate periods does not alter any of the structural results and insights.)

Physical or other operational limitations constrain the total quantity of input that may be procured and/or processed in any period. For example, the availability of labor, equipment etc. to handle inbound operations can impose restrictions on the quantity of input that can be procured. Similarly, the quantity of input that can be processed in a given period maybe limited by various factors. We model these by imposing a per-period procurement and processing capacity restriction.
of $K$ and $C$ units respectively. The firm incurs a variable cost of $p_n$ to process one unit of input into the output commodity in period $n$. Finally, for simplicity, we assume operational costs such as physical holding costs and handling costs for the various commodities are negligible (While this assumption may not be true for different commodity markets, including a physical holding cost or handling cost in the model does not alter any of the insights derived from the analysis that follows).

At the beginning of each period $n$, the firm observes the input spot price, $S_n$, and the output forward prices, $\vec{F}_n$. Let $e_n$ and $Q_n$ denote the input and output commodity inventories respectively at the beginning of period $n$. Let $\vec{R}_n = (R_{N1}^n, \ldots, R_{NL}^n)$ denote the vector of cumulative commitments at the beginning of period $n$ against each forward contract yet to mature. Based on this information, the firm makes the following decisions in each period: 1) the quantity of input commodity to procure, $x_n$, 2) the quantity to process, $y_n$ and 3) the quantity of the output commodity to commit to sale against the forward contracts, $\vec{z}_n = (z_{N1}^n, \ldots, z_{NL}^n)$.

The procurement and processing decisions in any period are subject to capacity and inventory availability constraints and the feasible set of actions in period $n$ is given by $A_n(e_n) = \{(x_n, y_n) : 0 \leq x_n \leq K, \ 0 \leq y_n \leq \min\{C, e_n + x_n\}\}$.

The output commodity sale commitments are not reversible; i.e., $\vec{z}_n \geq \vec{0}$. The firm can commit to sell more output than is available on-hand as long as all the output committed for sale against a forward contract is delivered on the delivery date specified in the forward contract. That is, for each $N_\ell$ we require $R_{N\ell-1}^n + z_{N\ell-1}^n \leq Q_{N\ell-1} + y_{N\ell-1}$. We denote the set of feasible commitment vectors in period $n$ by $Z_n(\vec{R}_n, Q_n, m_n)$. The state transition equations for the input inventory and cumulative output commitments are given by $e_{n+1} = e_n + x_n - y_n$ and $\vec{R}_{n+1} = \vec{R}_n + \vec{z}_n$ respectively, while the state transition for output inventory is given by

$$Q_{n+1} = \begin{cases} Q_n + y_n & n \neq N_\ell - 1 \text{ for any } \ell \\ Q_n + y_n - R_{N\ell}^n - z_{N\ell}^n & n = N_\ell - 1 \end{cases}$$

The operational cashflows realized by the firm in period $n$ are given by

$$\Pi_n(x_n, y_n, \vec{z}_n, \mathcal{I}_n) = \begin{cases} \sum_{l=1}^L \beta^{N_l - n} F_n^{N_l} z_{N_l}^n - S_n x_n - p_n y_n & n < N \\ S_N e_N & n = N \end{cases}$$

where $\beta$ is the risk-free discount factor.

The profit function in equation (7) above recognizes revenues from output sales at the time of commitment rather than at delivery. Because commodity sale commitments are not reversible and
we assume no counter party risk, discounting using the risk-free discount factor and recognizing revenue at the time of commitment rather than at delivery is without loss of generality.

We model the firm’s objective function in period $n$ by the time-consistent risk measure $DCVaR_{\vec{\eta},n}(\cdot)$. Assuming the firm uses the optimal financial trading policy to hedge operational cashflows generated by any given operational policy and the conditions of Theorem 1 hold, we can determine the optimal operational policy by solving the following SDP

$$V_n(e_n, Q_n, \bar{R}_n, \mathcal{I}_n) = \max_{(x_n, y_n) \in A_n(\cdot), \bar{z}_n \in \mathcal{Z}_n(\cdot)} \left\{ \Pi_n(x_n, y_n, \bar{z}_n, \mathcal{I}_n) + \beta \mathbb{E}_{\pi_n}\left[ CVaR_{\eta_{n+1}, n+1}(V_{n+1}(\cdot)) \right] \right\} \quad (8)$$

for $n < N$ and $V_N(e_N, Q_N, \bar{R}_N, \mathcal{I}_N) = S_N e_N$.

Notice that the objective function in equation (8) has similarities to the objective function in an expected value maximization formulation. However, there are crucial differences. Specifically, the expectation with respect to the risk-neutral probability measure is taken only over the states of market uncertainty. Further, the argument for the expectation operator is $CVaR_{\eta_{n+1}, n+1}(V_{n+1})$ of $V_{n+1}$.

In a complete market setting, $V_n$ is the unique market price of the operational assets (procurement and processing capacities). In the current setting, because markets are not complete there is no unique market price for the operational assets and the value assigned to these assets depends on the firm’s risk costs.

5. Operational Policy and Hedging
5.1. Optimal Operational Policy

In our model of partially complete markets, the uncertainty in revenues from the sale of a unit of output depends only on market uncertainty. As a result, the uncertainty can be perfectly hedged by trading in the financial markets and as Lemma 1 below states, the optimal commitment policy and the marginal value of output inventory are the same as those for a firm facing no risk costs (see Devalkar et al. 2011).

**Lemma 1.** It is optimal to postpone commitment against any specific contract $N_\ell$ to period $N_\ell - 1$. The optimal commitment decision in period $N_\ell - 1$ is then given by

$$z^{(N_\ell \times)}(n) = \begin{cases} Q_n + y_n - R_1^{N_\ell} & \text{if } n = N_\ell - 1 \text{ and } F_n^{N_\ell} \geq \mathbb{E}_{\pi_n}[\Delta_{n+1}^m] \\ 0 & \text{otherwise} \end{cases} \quad (9)$$

where $\Delta_n^m = \begin{cases} 0 & n \geq N_L \\ \beta \max\{F_n^{N_\ell}, \mathbb{E}_{\pi_n}[\Delta_{n+1}^m]\} & \text{if } n = N_\ell - 1 \text{ for } \ell = 1, \ldots, L \\ \beta \mathbb{E}_{\pi_n}[\Delta_{n+1}^m] & \text{otherwise} \end{cases} \quad (10)$


is the marginal value of a unit of output inventory.

Under an optimal commitment policy, \( R^{\ell N}_n = R^{1N}_1 = 0 \) for all \( n < N_\ell \), for all \( N_\ell \), so we do not need to keep track of the vector \( \tilde{R}_n \) as part of the state variable and \( V_n(Q_n, e_n, T_n) \triangleq V_n(Q_n, e_n, 0, T_n) = \Delta^m_n Q_n + U_n(e_n, T_n) \) for all \( n \leq N \), where

\[
U_n(e_n, T_n) = \max_{0 \leq y_n \leq \min\{C, e_n + x_n\}, \ 0 \leq x_n \leq K} \left\{ (\Delta^m_n - p_n) y_n - S_n x_n \right. \\
+ \beta \mathbb{E}_{\pi_n}\left[ CVaR_{\pi_n}^{(p)} \left( R_{(n+1),N} \right) \left( U_{n+1}(e_{n+1} + x_n - y_n, T_{n+1}) \right) \right]\right\}
\]

for \( n < N \) and \( U_N(e_N, T_N) = S_N e_N \).

While the optimal commitment policy is not affected by the risk costs, the total quantity of input procured and processed depend on the input spot price. Because the input spot price is affected by market and private uncertainties, the firm’s risk costs affect the procurement and processing decisions and thereby, the actual quantity of output committed for sale. As seen from equation (11), the firm’s optimal procurement and processing decision in period \( n < N \) is given by

\[
\arg\max_{0 \leq y \leq \min\{C, e_n + x\}, \ 0 \leq x \leq K} \left\{ (\Delta^m_n - p_n) y - S_n x + \beta \mathbb{E}_{\pi_n}\left[ CVaR_{\pi_n}^{(p)} \left( R_{(n+1),N} \right) \left( U_{n+1}(e_{n+1} + x - y, T_{n+1}) \right) \right]\right\}
\]

The firm needs to trade off the certain value from processing (and optimally committing to sell the output), \( \Delta^m_n - p_n \), with the risky option of carrying input inventory which can be processed in a later period or salvaged at the end of the horizon. The next theorem shows that the marginal value of input inventory is never more than the value from selling the output, provided the firm’s subjective probabilities on the salvage value \( S_N \) satisfy a mild regularity condition.

**Theorem 2.** Let the firm’s subjective probabilities on salvage value \( S_N \) be such that

\[
\beta^{N-(N_L-1)} \mathbb{E}_{\pi_{N_{L-1}}} \left[ CVaR_{\pi_{(N_L,N)}} (S_N) \right] < \Delta^m_{N_{L-1}} - p_{N_{L-1}} \text{ for each } T^m_{N_{L-1}}
\]

Then,

1. The value function \( U_n(e_n, T_n) \) is piecewise linear, concave and continuous in \( e_n \) with break points at integral multiples of \( D \), where \( D \) is the greatest common divisor of the procurement and processing capacities, \( K \) and \( C \), respectively.

2. The marginal risk-adjusted value of input inventory in period \( n \), denoted by \( \Theta^{(k)}_n \), when \( e_n \in [(k-1)D, kD) \) for any positive integer \( k \) is given by

\[
\Theta^{(k)}_n = \begin{cases} 
S_N & \text{for } n = N \text{ and } k \geq 1 \\
\max \left\{ \Omega^{(k+b)}_n, \min \left\{ S_n, \Omega^{(k)}_n \right\} \right\} & \text{for } n < N \text{ and } k \geq 1 
\end{cases}
\]

(13)
where \( \Omega_n^{(k)} = \max \{ \Upsilon_n^{(k)}, \min \{ \Delta_n^m - p_n, \Upsilon_n^{(k-a)} \} \} \)  

(14)

and \( \Upsilon_n^{(k)} = \left\{ \begin{array}{ll} \infty & \text{for all } n \text{ and } k \leq 0 \\ \beta \mathbb{E}_{\pi_n} \left[ CV aR^{(p)}_{(\eta_{n+1}, n+1)} \left( \Theta_{n+1}^{(k)} \right) \right] & \text{for } N_L - 1 < n < N \text{ and } k \geq 1 \\ \beta \mathbb{E}_{\pi_n} \left[ -CV aR^{(p)}_{(\eta_{n+1}, n+1)} \left( - \Theta_{n+1}^{(k)} \right) \right] & \text{for } n \leq N_L - 1 \text{ and } k \geq 1 \\ \end{array} \right. \)  

(15)

with \( a \) and \( b \) being integers such that \( C = aD \) and \( K = bD \).

The condition implied by inequality (12) says that the expected salvage value in the worst \( \eta \) fraction of cases is less than the value from processing the input and selling the output, for a given realization of market uncertainty. This is a reasonable assumption for a firm facing risk costs that would prefer the certain revenue from processing and selling the output rather than the possibly higher but uncertain revenue from salvaging. In fact, it is not unreasonable to assume that \( \beta^{N-(N_L - 1)} \mathbb{E}_{\pi_{N_L-1}} [ \mathbb{E}_{q_N} [S_N] ] < \Delta_{N_L-1}^m - p_{N_L-1} \); i.e., the expected salvage value based on the firm’s subjective probabilities is no more than the value from processing and selling the output. As \( CV aR^{(p)}_{(\eta_{N_L}, N_L)} (S_N) \leq \mathbb{E}_{q_N} [S_N] \) for all \( \eta \in (0, 1] \), the condition in equation (12) is less restrictive.

The results in Devalkar et al. (2011) are a special case of the results presented here (notice that \( \Upsilon_n^{(k)} \) in equation (15) reduces to the discounted expected marginal value of input inventory when markets are complete). We see from the proof of Theorem 2 that the optimal procurement and processing quantities are governed by price and horizon dependent thresholds and the optimal policy is as given in Corollary 1.

**Corollary 1.** The optimal procurement and processing quantities, \( x_n^* \) and \( y_n^* \), in period \( n \) are:

\[
x_n^* = \begin{cases} 
K & \text{if } \Omega_n^{(k+b)} \geq S_n \\
\hat{k}_n D - e_n & \text{if } \Omega_n^{(k)} \geq S_n > \Omega_n^{(k+b)} \\
0 & \text{if } S_n > \Omega_n^{(k)} 
\end{cases}
\]  

(16)

where \( \hat{k}_n = \max \{ k \in \mathbb{Z}^+ \text{ s.t. } \Omega_n^{(k)} > S_n \} \) and

\[
y_n^* = \begin{cases} 
0 & \text{if } \Omega_n^{(i)} > \Delta_n^m - p_n \\
\min \{ e_n + x_n^* - \hat{j}_n D, C \} & \text{if } \Omega_n^{(i)} \leq \Delta_n^m - p_n 
\end{cases}
\]  

(17)

where \( i \) is an integer s.t. \( e_n + x_n^* \in [(i-1)D, iD) \) and \( \hat{j}_n = \max \{ j \in \mathbb{Z}^+ \text{ s.t. } \Upsilon_n^{(j)} > \Delta_n^m - p_n \} \).
5.2. Impact of risk costs on operational policy

When markets are not complete, the risk costs faced by the firm have an impact on the marginal value-to-go of the input inventory and the thresholds governing the optimal procurement and processing decisions are a function of the firm’s risk costs (in addition to the horizon length and realized prices). In this section, we look at how the risk costs, measured by $\eta$, affects the optimal operational policy. Our main result, stated in Theorem 3 below, shows that the optimal procurement quantity increases with the risk costs.

**Theorem 3.** Let $\vec{\eta}$ and $\vec{\eta}'$ be two vectors of length $N$ such that $\eta_n \geq \eta'_n$ for all $n < N$ and $\eta_N = \eta'_N$, and $\eta_n, \eta'_n \in (0, 1]$ for all $n$. Then, for all $I_n$, for all $n < N$

1. The marginal value of input inventory, $\Theta^{(k)}_n$, when risk costs are represented by $\vec{\eta}'$ is no less than the marginal value of input inventory when risk costs are given by $\vec{\eta}$, for all $k$.

2. The optimal quantity of input procured, $x^*_n$, by a firm whose risk costs are represented by $\vec{\eta}'$ is no less than the optimal quantity of input procured by a firm whose risk costs are represented by $\vec{\eta}$, when both firms have the same starting input inventory $e_n$.

Notice that $\vec{\eta}'$ represents a higher level of risk costs, compared to $\vec{\eta}$. Theorem 3 states that for a given $e_n$ and $I_n$, the optimal procurement quantity is non-decreasing in the risk costs faced by the firm. Many existing results in the operations management literature about the impact of risk aversion suggest that the optimal procurement quantity decreases with risk aversion (see Eeckhoudt et al. 1995, for a newsvendor example). To see why the impact of increasing risk costs on optimal procurement is different in the current context, notice that the marginal value of inventory, $\Theta^{(k)}_{n+1}$, at the beginning of period $n + 1$ is the benefit of carrying an extra unit of input inventory from period $n$ to $n + 1$. In our model, the risk-adjusted benefit of carrying inventory is increasing in the risk costs. As a result, for a given realization of the input price, $S_n$, in the current period, a firm facing higher risk costs procures a larger quantity in period $n$. While there is no demand uncertainty, the procurement cost (and revenues) in future periods, however, are uncertain and as a result there is an implicit ‘cost of understocking’ $(\Theta^{(k)}_{n+1})$. Theorem 3 implies that the ‘cost of understocking’ is increasing in the risk costs, and therefore results in a higher quantity of the input being procured for the same ‘cost of overstocking’; i.e., current period procurement price $S_n$.

6. Numerical Study

From the analysis so far, we see that a firm facing commodity price uncertainty manages risk costs by adapting its operational and financial trading to the evolving uncertainty in the commodity prices. One of the articulated advantages of using a risk measure such as DCVaR is that it
accounts for the evolution of uncertainty over time in a consistent manner. However, the preceding analysis does not quantify the benefit from time-consistent decision making. For instance, what is the impact of using a time-consistent objective function on the cash flows realized in intermediate periods and the overall risk faced by the firm? Is the risk in terminal wealth reduced when the firm uses a time-consistent objective function? We conduct numerical studies to answer these questions and examine the benefits from time-consistent decision making. These results are presented in Section 6.2, after we discuss the details of numerical study implementation in Section 6.1.

The risk costs faced by the firm, quantified by $\vec{\eta}$, affect the optimal operational and financial trading policies when it maximizes a risk-adjusted objective function. The results of our numerical studies investigating the impact of risk costs on expected profits and procurement are presented in Section 6.3. In Section 6.4, we compare the performance of the optimal policy with that of common heuristics such as risk-neutral and myopic policies that firms might use to determine operational and financial trading decisions to manage risk costs.

### 6.1. Numerical study implementation

**Commodity price processes.** In our model, the input and commodity spot prices are a function of market uncertainty. In addition to market uncertainty, the input spot price is also affected by private uncertainty. We use a single factor, mean-reverting process outlined in Jaillet et al. (2004) to model the evolution of the market uncertainties affecting the input and output prices. Specifically, $S^m(t)$, the component of the commodity spot price driven by market uncertainty, is modeled as

$$\ln S^m(t) = \chi(t) + \mu(t),$$

where $\chi(t)$ is the logarithm of the deseasonalized price and $\mu(t)$ is a deterministic factor that captures the seasonality in spot prices. The deseasonalized price $\chi(t)$ follows a mean-reverting process given by

$$d\chi(t) = \kappa(\xi - \chi(t))dt + \sigma dW(t),$$

where $\kappa$ is the mean-reversion coefficient, $\xi$ is the long run log price level, $\sigma$ is the volatility and $dW(t)$ is the increment of a standard Brownian motion.

The parameters of the market price processes under the risk-neutral measure for the input and output commodities are estimated by calibrating them to the observed futures prices for the various commodities, as described in Jaillet et al. (2004). The futures price information on futures contracts traded on the Chicago Board of Trade (CBOT) for different maturities on each trading day of the month of June 2010 for soybean, soybean meal and soybean oil was used to calibrate the parameters. For the purposes of the numerical study in this paper, we model a single composite output, whose parameters are estimated by combining the prices of the soymeal and oil in the proportion in which they are produced upon processing one unit of the input commodity. Such approximations
Table 2  Price Process Parameters

(a) Deseasonalized price parameters

<table>
<thead>
<tr>
<th></th>
<th>Input (Soybean)</th>
<th>Output (Soymeal and oil Composite)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean-Reversion Coeff $\kappa_i$</td>
<td>0.229</td>
<td>0.535</td>
</tr>
<tr>
<td>Longrun Log level $\xi_i$</td>
<td>6.738</td>
<td>6.833</td>
</tr>
<tr>
<td>Volatility $\sigma_i$</td>
<td>0.244</td>
<td>0.436</td>
</tr>
</tbody>
</table>

(b) Seasonality Factor $e^{\mu(t)}$

<table>
<thead>
<tr>
<th></th>
<th>Jan / Feb</th>
<th>Mar / Apr</th>
<th>May / Jun</th>
<th>Jul</th>
<th>Aug</th>
<th>Sep</th>
<th>Oct</th>
<th>Nov</th>
<th>Dec</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input</td>
<td>0.992</td>
<td>0.998</td>
<td>1.000</td>
<td>1.017</td>
<td>1.010</td>
<td>0.991</td>
<td>0.991</td>
<td>0.989</td>
<td>0.989</td>
</tr>
<tr>
<td>Output</td>
<td>0.988</td>
<td>0.993</td>
<td>0.995</td>
<td>1.037</td>
<td>1.013</td>
<td>1.000</td>
<td>0.987</td>
<td>0.987</td>
<td>0.984</td>
</tr>
</tbody>
</table>

of multiple commodities as a single, composite product have been used in the context of valuing basket options (see Borovkova et al. 2007, for instance) and are sufficient to illustrate the main goals of the numerical study. The average of the estimated parameters obtained over each trading day are given in Table 2. The correlation between the Brownian motion increments underlying the input and output commodity market price processes, calibrated using historical weekly returns on the nearest maturing futures contracts, was estimated to be 0.883.

We model the private uncertainty affecting the input spot price as follows. For a given value $S^m(t)$ of the market price at time $t$, the spot price at which the firm can procure the input, $S(t)$ is given by $S(t) = S^m(t) \times b$, where $b$ is the basis risk (or basis return). We model $b$ as a log-normally distributed random variable with mean 1 and volatility $\sigma_p$; i.e., $b = \exp(- (\sigma_p)^2/2 + \sigma_p \varepsilon)$ where $\varepsilon \sim N(0,1)$. Further, $\varepsilon$ is independent of the market uncertainty and has no time increment associated with it. The dependence of the volatility of the input price on the time increment is captured through the market uncertainty process instead.

**Discretization of the price process.** We use the re-combining binomial tree procedure outlined in Peterson and Stapleton (2002) to discretize the dynamics of the continuous price processes described above and approximate the joint evolution of the market spot price of the input and output commodities. Each period in the discrete binomial tree corresponds to a week and we discretize the price process with $\delta = 10$ steps between each period. Thus, we have $[(n - 1)\delta + 1]^2$ nodes in the tree for period $n$, with each of these nodes corresponding to a given state of market uncertainty; i.e., each node corresponds to a $I^m_n$. At each node in the tree, we can compute the forward price $F_{n,l}$ for $l = 1, 2, \ldots, L$ for the output commodity and the component of the input spot price driven by market uncertainty, $S^m_n(I^m_n)$. For each node of market uncertainty, $I^m_n$, we approximate the contingent private uncertainty by discretizing $b$, the basis risk. We use $P = 100$ discrete values
to approximate the private uncertainty; i.e., for each $T_n^m$, we have 100 possible private states, $T_n^p$, each characterized by $b(T_n^p)$. The input spot price for a given state $T_n$, for $n < N$ is given by $S_n(T_n) = S_n^m(T_n^m) \times b(T_n^p)$ and for $n = N$, we set $S_N(T_N) = (\Delta_N^m(T_N^m) - p_N) \times b(T_N^p)$ to ensure the condition given by inequality (12) is satisfied.

**Other operational parameters.** We set the processing capacity $C$ to 3 units for each period, and consider different values of the procurement capacity, $K$. In the current context, the capacity can be considered to be in multiples of bushels, e.g., millions of bushels of soybean that can be processed in a period. Apart from assuming that both the procurement and processing capacities are in the same units, we leave the exact units unspecified. Throughout, we set the variable cost of processing $p$ to equal 72 cents / unit, that, based on the long run average prices of the commodities, corresponds to about 35% of the gross margin from processing one unit of soybean. This value of the processing cost is close to the average processing costs estimated for the US soybean processing industry (Soyatech 2008). We assume the physical holding costs for the various commodities are negligible and normalize them to zero and set the discount factor to 1.

**Computation of the optimal policy.** With the discrete prices, we use equations (10) and (13) to compute $\Delta_n^m$ and $\Theta_n^{(k)}$ and thereby the procurement and processing policy at each node in the tree. The computational time required to calculate the optimal policy parameters (Table 3) are not significantly different from the times reported in Devalkar et al. (2011) to compute the optimal risk-neutral policy.

### 6.2. Value of time-consistent decisions

In a partially complete market framework, the private states of uncertainty are not spanned by the financial securities traded in the market. As a result, cashflows that depend on the private states cannot be replicated exactly and there will be residual variability in cashflows across different states even when the firm undertakes financial hedging. The operational and financial trading policy used by the firm affects this variability in cashflows and the associated risk costs faced by the firm. It is therefore important to understand the benefits of using a time-consistent objective function such as DCVaR to the firm in terms of managing the risk costs associated with uncertain cashflows.

To quantify the value of using a time-consistent risk measure, we compare the performance of the firm when it maximizes a time-consistent risk measure (DCVaR) to the case when it maximizes...
a static risk measure (CVaR) on terminal wealth (or total profits earned over the horizon), with both risk measures being evaluated for different values of $\eta$. The optimized value of the objective function under the two criteria, however, are not directly comparable (please see Appendix B for details). Therefore, we need some common metrics to compare performance. A natural comparison is to consider the total accumulated cashflows over the horizon, $W_{N+1}$, generated by the policies that maximize DCVaR and CVaR respectively, for a given value of $\eta$. In addition to total cashflows, uncertainty in cashflows in intermediate periods is an important consideration while managing risk over a multi-period horizon. Therefore, we also compare the minimum accumulated cashflows at any point in the horizon, $W_{\min}$, generated by the different policies.

Because $W_{N+1}$ and $W_{\min}$ are random variables under both the DCVaR and CVaR criteria, we need a way to quantify their riskiness. We use CVaR of $W_{N+1}$ and $W_{\min}$ as a metric to compare performance for the following reasons. First, CVaR is the basic single-period risk measure used in the construction of our time-consistent measure. Second, as shown in Dentcheva and Ruszczyński (2006), for two random variables $X$ and $Y$ if $CVaR_{\alpha}(X|\mathcal{I}) \geq CVaR_{\alpha}(Y|\mathcal{I})$ for all $\alpha \in (0,1]$, then $X$ stochastically dominates $Y$ in the second order sense. We compare $CVaR_{\alpha}(W_{N+1}|\mathcal{I}_1)$ and $CVaR_{\alpha}(W_{\min}|\mathcal{I}_1)$ for a range of values $\alpha \in (0,1]$, for each policy.

As mentioned in section 6.1, the price processes are discretized and therefore using either CVaR of terminal wealth or DCVaR criterion, the optimization problem can be represented as a linear program. Representing the evolution of the commodity prices as a tree allows us to enumerate all the sample paths and obtain a distribution of $W_{N+1}$ and $W_{\min}$ for each policy. However, this procedure is computationally intensive, especially for the CVaR criterion, and therefore we can only show results for a small horizon problem. The results reported in this section were generated for a 4 period horizon using the parameters given in Table 2 and setting $\delta$ and $\mathcal{P}$ equal to 2. While we do not report it, we found similar results for different combinations of price volatilities and different starting processing margin values. Figure 1 shows the $CVaR_{\alpha}(W_{\min}|\mathcal{I}_1)$ profile under the DCVaR and CVaR policies for $\eta = 0.05$ and $\eta = 0.55$, to illustrate the impact on risk in intermediate periods under the two objective functions. As seen in the figure, using a time-consistent risk measure greatly reduces the risk in intermediate periods: in fact, in all our numerical studies, the distribution of the minimum wealth over the horizon when DCVaR is maximized, first order stochastically dominates the distribution of minimum wealth over the horizon when CVaR of terminal wealth is maximized. This suggests that accounting for uncertainty resolution in a consistent manner enables better control over risk costs all through the planning horizon.
It is worth noting that better control over risk in intermediate periods when maximizing DCVaR comes with little sacrifice in the distribution of total wealth, as seen from Figure 2. When risk costs are high ($\eta = 0.05$), both policies are conservative. As Figure 2(a) shows, the DCVaR policy is much more conservative and loses out on upside opportunities that the CVaR policy exploits. The expected profits, value of $CVaR_\alpha(W_{N+1}|I_1)$ for $\alpha = 1$, is 33 for the CVaR policy as compared to 25.71 for the DCVaR policy. The cost of this increased expected value of profits is the higher risk in intermediate periods, as seen in Figure 1(a). When risk costs are moderate ($\eta = 0.55$), both policies are willing to take some risks to take advantage of upside opportunities. This is reflected by the higher expected profits and lower values of $CVaR_\alpha(W_{N+1}|I_1)$ small $\alpha$. In the case of the CVaR policy, this increase in expected profits comes with very high risk not just in intermediate periods but also in total wealth. To put the performance of the two policies in contrast, the expected profits and $CVaR_\alpha(W_{N+1}|I_1)$ for $\alpha = 0.05$ under the DCVaR policy are 717.45 and $-6.39$ respectively, whereas the corresponding values under the CVaR policy are equal to 603.52 and $-1076.34$ respectively. In Figure 2(b), notice that $CVaR_\alpha(W_{N+1}|I_1)$ with the CVaR criterion dominates in a small neighborhood around the region $\alpha = \eta$. This is to be expected because the CVaR objective function maximizes $CVaR_\eta(W_{N+1}|I_1)$. However, at other values of $\alpha$, the DCVaR criterion results in a higher value of $CVaR_\alpha(W_{N+1}|I_1)$.

The above numerical study illustrates the benefits of using a time-consistent objective function such as DCVaR in terms of managing risk throughout the planning horizon. Compared to using a time-inconsistent objective function such as CVaR, DCVaR also has computational advantages due smaller state space requirements. While computing the optimal DCVaR policy requires current inventory and price information, computing the optimal CVaR policy requires additional information of the current wealth and distribution of total profits over all future periods.
6.3. Impact of risk sensitivity on performance

In this section, we consider how risk costs faced by the firm, quantified by $\bar{\eta}$, affect the total expected profits and the total expected quantity procured over the horizon. For the purposes of this study, with processing capacity, $C$, set to 3 units per period, we consider different levels of the procurement capacity $K$. While we show results for scenarios where $K = C + 1$ and $K = C + 2$, the results for other values of $K$ and $C$ are qualitatively similar to the results presented here. In all the studies, we assume zero starting input inventory.

Input price risk affects the firm in two ways. During the season, any increase in input prices increases the cost of procurement whereas at the end of the season any decrease in input prices reduces the value from trading the input. The firm’s overall risk sensitivity factors both the risk of higher input procurement prices, and lower input trade (or salvage) prices. While not an exact measure, $\eta_N$, the firm’s risk sensitivity parameter in the last period, captures the firm’s sensitivity towards loss in value from trading the input. We use the term ‘trade price risk sensitivity’ to denote $\eta_N$. Similarly, the firm’s risk sensitivity parameters in the intermediate periods, $(\eta_1, \ldots, \eta_{N-1})$, capture the firm’s sensitivity to higher procurement costs. We use the term ‘procurement price risk sensitivity’ to denote $(\eta_1, \ldots, \eta_{N-1})$ and for the purposes of this study, set $\eta_1 = \ldots = \eta_{N-1} = \eta(1, N-1)$.

Theorem 3 says that for the same level of trade price risk sensitivity, the total input procured for any given state $(e_n, I_n)$ in period $n$ is non-decreasing in the firm’s procurement price risk sensitivity. However, by procuring more in period $n$, the firm is likely to carry more input inventory, $e_{n+1}$, at the beginning of the next period. Because the marginal value of input inventory is non-increasing in the current inventory level, it is possible that the firm finds it optimal to procure a lesser quantity of input in period $n + 1$. Thus, it is not clear how the total quantity procured over
the horizon itself changes as the firm’s procurement price risk sensitivity increases.

Figure 3 shows the expected procurement quantity as a function of the procurement price risk sensitivity, for different levels of trade price risk sensitivity, for planning horizons of length 5 and 10 periods, when the firm uses optimal operational and financial hedging policies. The x–axis shows the sensitivity to risk costs during the horizon as measured by $(1 - \eta(1,N-1))$, where $\eta(1,N-1)$ is the probability level at which $CVaR_\eta(p_{n,n})$ is evaluated in each period $n < N$. Risk sensitivity during the horizon increases from left to right on the x–axis, with $(1 - \eta(1,N-1)) = 0$ representing a firm that is not sensitive to procurement price risk (but may still be sensitive to trade price risk).

From the figure, it appears that the expected input procurement increases with the firm’s procurement price risk sensitivity for a given level of input trade price risk sensitivity. It is not surprising that as $\eta_N$ increases, the firm’s expected procurement quantity increases for all levels of procurement price risk sensitivity. This is because the risk-adjusted salvage price increases with $\eta_N$.

![Figure 3](image1.png)

(a) $N = 5$, $L = 1$, $N_1 = 5$  (b) $N = 10$, $L = 2$, $(N_1, N_2) = (5, 10)$

**Figure 3** Impact of procurement price risk sensitivity on input procurement ($K = C + 1$)

Figure 4 shows how the expected total profit changes with the firm’s procurement price risk sensitivity. Figure 4(a) shows that the expected profits are not necessarily monotonic in the procurement price risk sensitivity. This is because the excess input procured at different $\eta(1,N-1)$ levels are at different input spot prices and each unit may not necessarily result in positive margin when traded at the end of the horizon. As a result, the total expected profits is not necessarily monotonic. As one would expect, the expected total profit increases as the trade price risk sensitivity decreases ($\eta_N$ increases) for all levels of procurement price risk sensitivity.

**Impact of overall risk.** Figures 3 and 4 show the impact of procurement price risk sensitivity on the firm’s procurement and profits for a given value of $\eta_N$. As such, they do not provide an insight into how the firm’s overall sensitivity to risk costs affects performance.
From equation (12), we can see that it is always beneficial to process any input and sell it as output rather than trading it as input at the end of the horizon. As a result, when $K = C$, input inventory will never be carried over and the profit in any period is determined completely by the processing margin. Further, the revenues from processing and selling the output are completely determined by the market uncertainty, and the firm procures input only when the realized input spot price is less than the revenue from processing and selling the output. Thus, when $K = C$, the firm is effectively following a myopic operational policy, where it procures and processes up to capacity if and only if there is a non-negative processing margin; i.e., $\Delta m_n - p_n - S_n \geq 0$. 

Figures 5 and 6 show the total expected input procurement and the total expected profits, respectively, as a function of the degree of risk costs, for different values of procurement capacity. The risk costs increase from left to right along the x-axis, with $(1 - \eta) = 0$ representing a risk-neutral firm where $\eta$ is the probability level at which $CVaR_{\eta n}(p)$ is evaluated in each period $n$. 

From equation (12), we can see that it is always beneficial to process any input and sell it as output rather than trading it as input at the end of the horizon. As a result, when $K = C$, input inventory will never be carried over and the profit in any period is determined completely by the processing margin. Further, the revenues from processing and selling the output are completely determined by the market uncertainty, and the firm procures input only when the realized input spot price is less than the revenue from processing and selling the output. Thus, when $K = C$, the firm is effectively following a myopic operational policy, where it procures and processes up to capacity if and only if there is a non-negative processing margin; i.e., $\Delta m_n - p_n - S_n \geq 0$. 

Figure 4  Impact of procurement price risk sensitivity on total profits ($K = C + 1$) 

Figures 5 and 6 show the total expected input procurement and the total expected profits, respectively, as a function of the degree of risk costs, for different values of procurement capacity. The risk costs increase from left to right along the x-axis, with $(1 - \eta) = 0$ representing a risk-neutral firm where $\eta$ is the probability level at which $CVaR_{\eta n}(p)$ is evaluated in each period $n$. 

From equation (12), we can see that it is always beneficial to process any input and sell it as output rather than trading it as input at the end of the horizon. As a result, when $K = C$, input inventory will never be carried over and the profit in any period is determined completely by the processing margin. Further, the revenues from processing and selling the output are completely determined by the market uncertainty, and the firm procures input only when the realized input spot price is less than the revenue from processing and selling the output. Thus, when $K = C$, the firm is effectively following a myopic operational policy, where it procures and processes up to capacity if and only if there is a non-negative processing margin; i.e., $\Delta m_n - p_n - S_n \geq 0$. 

Figure 5  Impact of overall risk sensitivity on input procurement
As the cost of risk \((1 - \eta)\) increases, not only is the firm’s procurement price risk sensitivity increasing, the trade price risk sensitivity is also increasing. As a result, the firm’s input procurement, over and above what is required for processing needs, for trading at the end of the horizon decreases. Consequently, when \(K > C\), the expected procurement and profits decrease with the risk costs faced by the firm as seen in figures 5 and 6. We can see from the figures that as risk costs increase from \(1 - \eta = 0\) to \(1 - \eta = 0.9\), the flow rate (measured by total expected input procurement) and financial value from operations (measured by expected value of cashflows under the risk-adjusted probability measure) decrease by 18% and 13% respectively. Comparing the total expected profits for different values of \(K\) in Figure 6, we see that regardless of the risk costs faced, additional procurement capacity is valuable for the firm with each additional unit of capacity contributing up to 20% increase in financial value even for firms with very high risk costs.

### 6.4. Performance of heuristics

In this section, we propose and evaluate the performance of two heuristics. The first heuristic we evaluate is a myopic policy that a firm facing significant risk costs is likely to use, because this heuristic will ensure non-negative operational cash flows in all states of the world. Using this heuristic, the firm procures the input, and processes all the available input, if and only if the revenue from processing and selling the output is greater than the cost of procurement; i.e., \(x_n = m_n = \min\{K, C\}\) if \(\Delta^m_n - p_n \geq S_n\) and \(x_n = m_n = 0\) otherwise. The second heuristic that we evaluate is the optimal risk-neutral policy, which is likely to be used by a firm that ignores its risk costs (or, does not assess these costs to be significant). In addition to operational decisions, the heuristics also specify the firm’s financial hedging decisions. For both the myopic and risk-neutral heuristics, the firm’s financial trading decisions replicate the expected value of operational cashflows over all private states corresponding to a given market state. Suppose \(X_{n+1}(I_{n+1})\) is the operational
cashflow generated by the heuristic in period \( n + 1 \), in state \( I_{n+1} \). Then, the firm trades a portfolio of financial instruments in period \( n \) such that the payoff from the portfolio in period \( n + 1 \) is equal to \( \mathbb{E} \left[ X_{n+1}(I_{n+1})|I_{n+1}' \right] \), for each \( I_{n+1}' \). In contrast, under the DCVaR-optimal policy, the firm would trade a portfolio such that the payoff in period \( n + 1 \) is equal to \( \text{CVaR}(\eta_n, n+1) (Y_{n+1}(I_{n+1})) \), for each \( I_{n+1} \), where \( Y_{n+1}(I_{n+1}) \) is the operational cashflow generated in period \( n + 1 \), in state \( I_{n+1} \) by the DCVaR-optimal operational policy.

In order to account for the risk costs faced by the firm when using the heuristic policies, we need to compare the resulting \( \text{DCVaR}(\bar{\eta}, 1) \) under each of these policies. Figure 7 shows \( \text{DCVaR}(\bar{\eta}, 1) \) for different values of \( 1 - \eta \) under the optimal and myopic policies, while Figure 8 compares the performance of the optimal and risk-neutral policies.

A firm using a myopic policy forgoes potential profits by not procuring enough input in states where procuring additional input would have been profitable. In other words, the myopic policy
assigns a marginal risk-adjusted value of zero for any input inventory that is carried forward to the next period. As seen from Figure 7, the value of these lost profit opportunities is quite high and of the order of 40 to 50% for firms facing low risk costs and are likely to increase as the procurement capacity increases relative to the processing capacity. As the risk costs \((1 - \eta)\) increase, the marginal risk-adjusted value of input inventory under the optimal policy decreases and we expect the operational cashflows generated by the optimal and myopic policies to be similar. However, because of a sub-optimal financial trading policy, the variability in total cash flows across market states is higher under the myopic policy compared to the optimal policy. Thus, the \(DCVaR(\vec{\eta}, 1)\) of total cashflows from using a myopic policy is significantly lower than the \(DCVaR(\vec{\eta}, 1)\) of total cashflows under the optimal policy.

When a firm faces no risk costs \((1 - \eta = 0)\), the DCVaR-optimal policy coincides with the optimal risk-neutral policy and both policies yield the same value. As risk costs \((1 - \eta)\) increase, the risk-neutral policy would be more aggressive compared to the optimal operational policy. As a result, the firm is likely to procure a higher quantity of input and also potentially procure even in states with higher input price realizations. As a result, the firm is exposed to higher risk, resulting in a lower \(DCVaR(\vec{\eta}, 1)\) value. As intuition would suggest, and also seen in figure 8, the negative consequences of following a risk-neutral policy are especially pronounced for firms facing high risk costs. In addition, the sub-optimal risk-neutral financial trading policy leads to higher variability in total cashflows across market states compared to the optimal policy.

In summary, our numerical results demonstrate that risk management considerations have a significant impact on a firm’s performance. We find that incorporating risk costs in a time-consistent manner result in better risk-reward outcomes and firms facing risk costs procure more as their procurement price risk sensitivity increases. Further, myopic operational policies, even when they are combined with optimal financial hedging, lead to a significant loss in value and heuristics that ignore risk costs and use policies that are optimal for a risk-neutral firm perform poorly as substitutes to the optimal policy.

7. Conclusions
In this paper we considered the dynamic financial and operational decisions for a commodity processing firm operating in a partially complete financial market. Using a time-consistent risk measure to model the firm’s risk management objective, we characterized the optimal financial trading and operational policies. We showed that the optimal financial trading strategy replicates
the CVaR (over private states of uncertainty) of the operational cashflows, for each market state, while the optimal operational policies are structurally similar to the optimal operational policies for a risk-neutral firm. However, the firm’s risk management objective has a significant impact on the parameters of the optimal policy and the value of a stream of operational cashflows is different for a firm that faces risk costs. Our numerical studies showed the value of optimizing a time-consistent risk management objective function. We also demonstrated that using the optimal risk-neutral policy or a myopic policy as heuristics can lead to substantial loss in value.

Our work is one of the few early attempts to model the dynamic risk management problem for firms dealing with storable commodities. The model presented here can be extended to consider a situation where the price of the output commodity is also affected by private uncertainty. Under suitable restrictions on the output price process (when longer maturity forward prices are lower than earlier maturity prices, that is, output forward prices exhibit normal backwardation), one can show that the optimal commitment policy is to commit all available output to the closest maturity forward. While the structure of the optimal procurement and processing policy is similar to the policy described in corollary 1, computing the procure-up-to and process-down-to thresholds is considerably more complicated in this case. The context of the problem can be expanded to include multiple input / output commodity sets where the firm has a choice to decide which input to procure and process, or what output (or mix of outputs) to produce. In the current paper, the uncertainty in cashflows is due to commodity prices. It will be interesting to study the risk management problem for a firm that faces uncertain demand for the output and/or input, in addition to price uncertainty. An important aspect of our model is the partially complete market framework where the realized private uncertainty does not convey information about future events. The analysis in the current paper can be extended to situations where market incompleteness is modeled more generally.

References


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Appendix A: Summary of frequently used notation

\[ I_n^m \]  
State of market information in period \( n \)

\[ I_n^p \]  
State of private information in period \( n \)

\[ I_n = (I_n^m, I_n^p) \]  
State of information in \( n \)

\[ \hat{M}_n(I_n^m) = (M_0^m(I_n^m), \ldots, M_J^m(I_n^m)) \]  
Vector of prices of the \( J + 1 \) market securities at the beginning of period \( n \)

\[ \beta = 1/(1 + r_f) \]  
The risk-free discount rate, where \( r_f \) is the risk-free interest rate

\[ \pi_n(I_{n+1}^m|I_n^m) \]  
Risk-neutral probability that market state in period \( n + 1 \) is \( I_{n+1}^m \), conditional on \( I_n^m \)

\[ \gamma_n(I_{n+1}^m|I_n^m) \]  
Firm’s subjective probability that market state in period \( n + 1 \) is \( I_{n+1}^m \), conditional on \( I_n^m \)

\[ q_n(I_n^p|I_n^m) \]  
Firm’s subjective probability that private state in period \( n \) is \( I_n^p \), conditional on market state in period \( n \) being \( I_n^m \)

\[ CVaR_\eta(X|I) \]  
CVaR of random variable \( X \) at the \( \eta \)-level, conditional on information available \( I \)

\[ CVaR^{(p)}(X_n(I_n)) = CVaR_\eta(X_n(I_n)|I_n^m) \]  
CVaR of \( X_n(I_n) \) at the \( \eta \)-level, conditional on the state of market information \( I_n^m \) in period \( n \)

\[ CVaR^{(m)}(X_{n+1}^m(I_n^m)) = CVaR_\eta(X_{n+1}^m(I_n^m)|I_n^m) \]  
CVaR of \( X_{n+1}^m(I_n^m) \) at the \( \eta \)-level, conditional on the state of market information \( I_n^m \) in period \( n \)

\[ \vec{\alpha}_n = (\alpha_0^m, \ldots, \alpha_J^m) \]  
Vector of positions in the \( J + 1 \) market securities at the end of period \( n \)

\[ S_n(I_n) \]  
Input commodity spot price in period \( n \), conditional on state of information \( I_n \)

\[ F_{n}^{N_k}(I_n^m) \]  
Forward price for output commodity in period \( n \) on contract maturing in period \( N_k > n \), conditional on \( I_n^m \)

\[ K \]  
Input procurement capacity per period

\[ C \]  
Input processing capacity per period

\[ p_n \]  
Variable cost of processing a unit of input in period \( n \)

\[ e_n \]  
Input inventory at the beginning of period \( n \)

\[ Q_n \]  
Output inventory at the beginning of period \( n \)

\[ \vec{R}_n = (R_{n}^{N_k}, \ldots, R_{n}^{N_L}) \]  
Vector of cumulative commitments for sale against forward contracts \( N_k, \ldots, N_L \) at the beginning of period \( n \)

\[ x_n \]  
Quantity of input procured in period \( n \)

\[ y_n \]  
Quantity of input processed in period \( n \)

\[ \vec{z}_n = (z_{n}^{N_k}, \ldots, z_{n}^{N_L}) \]  
Vector of output quantity committed for sale against forward contracts \( N_k, \ldots, N_L \) in period \( n \)
Appendix B: Risk Measures and Time Consistency in Multi-Period Settings

B.1. Coherent Risk Measures

A risk measure is a mapping from the set of all risks, e.g., random cashflows whose realizations are contingent on the state of the world at the end of the period, to the set of real numbers. Formally, let $\mathcal{X}$ denote the set of $\mathcal{F}$-measurable functions, where $\mathcal{F}$ denotes the state of the world at the end of the period. Then, $\rho : \mathcal{X} \to \mathbb{R}$ is a risk measure. For instance, $\mathbb{E}[X]$ and $\mathbb{V}[X]$, the expectation and variance respectively, of $X \in \mathcal{X}$ are simple examples of risk measures. Another risk measure that is commonly used in the financial industry is the Value-at-Risk (VaR) measure (Duffie and Pan 1997).

Given the difficulty in modeling risk costs, firms use risk measures as a proxy to assess these costs. Thus, when the firm has to choose between random outcomes, e.g., projects that lead to different cashflows in different states of the world, evaluating the risk measure of the different outcomes provides a means to rank the outcomes. Further, when the cashflows are a function of the firm’s decisions, the firm can solve an optimization problem, using a risk measure to model the risk costs, to determine the best course of action.

To formalize this, suppose $X(a)$ denotes the random cashflow whose distribution is a function of the firm’s decision $a$ that can be chosen from some feasible set $\mathcal{A}$. Then, we can model the firm’s optimization problem as

$$\max_{a \in \mathcal{A}} \{ \mathbb{E}[X(a)] \} \text{, s.t. } \rho(X(a)) \geq B$$

where $\mathbb{E}[\cdot]$ is the expectation operator, $\rho(\cdot)$ is a risk measure used to approximate the risk costs and $B$ is a scalar that represents a threshold on the risk costs the firm can bear. (Alternately, because we are defining risk measures on cashflows / profits, we can think of $-\rho(X(a))$ as representing the risk costs and the constraint on risk costs as $-\rho(X(a)) \leq -B$.)

Using Lagrange multipliers, we can modify the firm’s objective function as $\mathbb{E}[X(a)] + \gamma \rho(X(a))$, where $\gamma > 0$ is the scalar Lagrange multiplier, and eliminate the constraint. In a sense, we can think of $\mathbb{E}[\cdot] + \gamma \rho(\cdot)$ as the risk-adjusted value that the firm would like to maximize. A well known example of this approach of optimizing risk measures is the mean-variance framework for optimal portfolio selection (Markowitz 1952). The multiplier $\gamma$ captures the tradeoff between expected value and risk costs (Usually, risk measures are used in conjunction with random variables that measure negative consequences (losses), but can be used as mappings of cashflows / profits to the real line as well, as we do in this paper).

Unfortunately, not all popularly used risk measures are well behaved in the sense that using these risk measures may not lead to results that seem intuitively correct. For instance, it is intuitive that diversification should lead to lower risk; i.e., the riskiness of the sum of two cashflows should be no more than the sum of riskiness of each of the individual cashflows themselves. But the VaR risk measure, which is commonly used, does not always satisfy this property. As a result, several refinements have been proposed to overcome
these shortcomings and increase the appeal of risk measures as a good practical approach to model the objective function for a firm facing risk costs. In a seminal paper, Artzner et al. (1999) proposed the theory of coherent risk measures and introduced a class of risk measures that satisfy properties desirable from the perspective of decision making. Formally, if $\rho(\cdot)$ is a coherent risk measure, then it satisfies the following properties:

1. Monotonicity: For all $X$ and $Y \in \mathcal{X}$ such that $X \succeq_Y Y$, we have $\rho(X) \geq \rho(Y)$.
2. Sub-additivity: For all $X_1$ and $X_2 \in \mathcal{X}$, $\rho(X_1 + X_2) \geq \rho(X_1) + \rho(X_2)$.
3. Positive homogeneity: For all $\lambda \geq 0$ and $X \in \mathcal{X}$, we have $\rho(\lambda X) = \lambda \rho(X)$.
4. Translation invariance: For all $X \in \mathcal{X}$ and all real numbers $m$, we have $\rho(X + m) = \rho(X) + m$.

(The definitions and properties in Artzner et al. (1999) are based on risk measures defined on losses, while we model risk measures on profits or net cashflows. Therefore the properties of coherent risk measures stated above are appropriately modified.)

In addition to satisfying the properties mentioned above, one can use coherent risk measures to model the objective of a firm facing risk costs, instead of a mean-risk function, without loss of generality. To see this, let the objective function for a firm facing risk costs be written as $E[X] + \gamma \rho(X)$, where $E[\cdot]$ is the expectation operator, $\rho(\cdot)$ is a coherent risk measure used to approximate the risk costs, and $\gamma \geq 0$ is a scalar which captures the tradeoff between expected value and risk costs. Notice that,

$$E[X] + \gamma \rho(X) = (1 + \gamma) \left[ \frac{1}{1+\gamma} E[X] + \frac{\gamma}{1+\gamma} \rho(X) \right] = (1 + \gamma) [\alpha E[X] + (1 - \alpha) \rho(X)]$$

where $\alpha = 1/(1 + \gamma) \in (0, 1]$. It is easy to show that a convex combination of two coherent risk measures is also a coherent risk measure, and because $E[\cdot]$ is a coherent risk measure, the mean-risk objective function for the firm can be represented as $\hat{\rho}(X) = \alpha E[\cdot] + (1 - \alpha) \rho(\cdot)$. Thus, for the purposes of analysis, it is without of loss of generality to model the firm’s objective using a coherent risk measure rather than an explicit mean-risk function.

**B.2. Time consistency of Multi-period Risk Measures**

In a multi-period setting, where uncertainty is resolved over multiple periods, let $\mathcal{I}_n$ denote the information at the beginning of period $n$, and $\mathcal{X}_n$ be the set of $\mathcal{I}_n$-measurable functions, for $n = 1, \ldots, N$. For a multi-period cash flow stream $X = (X_1, \ldots, X_N)$, with $X_n \in \mathcal{X}_n$, we could define a dynamic risk measure as a sequence of coherent risk measures $\left( \rho_n \left( \sum_{\tau=1}^{N} X_\tau \right) \right)_{n=1}^{N}$ with $\rho_n : \mathcal{X}_N \rightarrow \mathcal{X}_n$, for each $n$.

We say a dynamic risk measure $\left( \rho_n(\cdot) \right)_{n=1}^{N}$ is time-consistent if for two random cashflows $X$ and $Y \in \mathcal{X}_N$, where $\mathcal{X}_N$ is the set of $\mathcal{I}_N$-measurable functions,

$$\rho_{n+1}(X) \geq \rho_{n+1}(Y) \quad \forall \mathcal{I}_{n+1} \Rightarrow \rho_n(X) \geq \rho_n(Y)$$

(18)

Time consistency conditions similar to or same as (18) have been studied in various contexts in the mathematical finance literature (see Roorda and Schumacher 2007, for instance). Condition (18) implies...
that if a cashflow stream that results in total value $X$ at the end of the horizon is ranked higher than a cashflow stream that results in total value $Y$ in every possible state of the world in period $n+1$, then it should be ranked higher in period $n$ as well. Somewhat surprisingly, not all applications of single period risk measures to dynamic settings ensure that the resulting dynamic measures are time-consistent. The following example, from Roorda and Schumacher (2007), illustrates the time inconsistency issue with risk measures.

**Example 1.** Suppose an operational investment, e.g., inventory investment, at the beginning of period 1 pays off at the end of period 2 (In this example, $X_1(I_1) = 0$ for all $I_1$). The payoffs are dependent on the state of the world in periods 1 ($u$ or $d$) and 2 ($u$, $m$ or $d$), and the probabilities for the different states of the world are as shown on the branches of the tree in Figure 9.

Suppose the firm uses $(CVaR_\eta(X_1 + X_2|I_n))^{n=1}_{n=0}$ at $\eta = 0.1$ as the dynamic risk measure to evaluate the investment. Evaluated at the beginning of period 1, the total payoff $X_1 + X_2$ is equal to $-20$, $-10$, and 12 with probabilities $0.5 \times 0.05 = 0.025$ each, and the next worst payoff is equal to 14 with probability $0.5 \times 0.9 = 0.45$. Thus, the expected payoff in the worst $10\%$ of cases as evaluated at $n = 1$ is $(0.025 \times (-20) + (-10) + 12) + (0.1 - (0.025 + 0.025 + 0.025)) \times 14)/0.1 = -1$. That is, $CVaR_{0.1}(X_1 + X_2|I_0) = -1$.

At the beginning of period 2, if $I_1 = u$, then the total payoff $X_1 + X_2$ is equal to $-10$ and 12 with probability 0.05 each, in the worst 0.1 cases. Thus, $CVaR_{0.1}(X_1 + X_2|I_1 = u) = (0.05 \times (-10) + 0.05 \times 12)/0.1 = 1$. Similarly, we have $CVaR_{0.1}(X_1 + X_2|I_1 = d) = (0.05 \times (-20) + 0.05 \times 22)/0.1 = 1$.

Compared to the option of doing nothing (which has a $CVaR_{0.1}(X_1 + X_2|I_{n-1}) = 0$ for all $n$), the above evaluation suggest that the project is acceptable in all states of the world at the beginning of period 2. However, the same project is not acceptable (compared to the option of doing nothing) in period 1. Notice that there are no cashflows occurring between periods 1 and 2 and there is no discounting. Thus, using a dynamic risk measure such as $(CVaR_\eta(X_1 + X_2|I_n))^{n=1}_{n=0}$ can lead to inconsistent decisions over time.

One approach to extend single period risk measures to a dynamic, time-consistent risk measure is the concept of conditional risk mappings (cf. Ruszczyński and Shapiro 2006). Conditional risk maps incorporate uncertainty resolution in a consistent manner and lead to a Bellman-equation-type structure for the dynamic risk measure. In fact, Cheridito and Kupper (2011) show that any time-consistent dynamic risk

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**Figure 9** Two period investment $X$
measure \((\rho_n(\cdot))_{n=1}^N\) where each individual \(\rho_n(\cdot)\) is a coherent risk measure, is uniquely given by a set of single period, coherent risk maps \(\psi_n: \mathcal{X}_{n+1} \to \mathcal{X}_n, n = 1, \ldots, N-1\) such that

\[
\rho_n(X) = \psi_n(\rho_{n+1}(X))
\]

Using this approach we can define a dynamic, time-consistent risk measure, labeled DCVaR, based on the single-period CVaR measure, such that \(DCVaR_n(X) \triangleq DCVaR(X|I_{n-1}) = CVaR_q(DCVaR_{n+1}(X)|I_{n-1})\). It is easy to verify that \(DCVaR_n\) is coherent for each \(n\), and satisfies the time consistency condition 18. For the payoff \(X_1 + X_2\) given in example 1, we have \(DCVaR_1(X_1 + X_2) = CVaR_0,1(X_1 + X_2|\mathcal{I}_2) = X_1 + X_2\), for each realization of \(\mathcal{I}_2\). Then, \(DCVaR_2(X_1 + X_2) = CVaR_0,1(DCVaR_1(X_1 + X_2)|\mathcal{I}_1)\) is CVaR_0,1(X_1 + X_2|\mathcal{I}_1) = 1 for \(\mathcal{I}_1 = u\) and \(d\). Finally, we have \(DCVaR_1(X_1 + X_2) = CVaR_0,1(DCVaR_2(X_1 + X_2)|\mathcal{I}_0) = 1\) because \(DCVaR_2(\cdot)\) is equal to 1 for all \(\mathcal{I}_1\). Even though the \(\eta\) levels when computing \(DCVaR_n(\cdot)\) and \(CVaR_{\eta,\cdot}(\cdot|\mathcal{I}_{n-1})\) are both equal in the above example, these levels are not directly comparable except for \(n = N\). This is because the \(CVaR_{\eta,\cdot}(\cdot)\) evaluations are done in a recursive fashion and the argument is not the sum of cashflows but the ‘risk adjusted’ value of the remaining cashflows, when computing \(DCVaR_n(\cdot)\).

Appendix C: Proofs of Theorems and Lemmas

Proof of Theorem 1 As there are no operational cashflows after period \(N\), the optimization problem in period \(N\) reduces to \(\max_{\mathcal{I}_N} \left\{ CVaR_{(\eta_N,N)}^r \left( \beta((\tilde{\alpha}_N)^T\tilde{M}_{N+1}) \right) - (\tilde{\alpha}_N)^T\tilde{M}_N \right\} \). Using equation (1), this reduced maximization problem over \(\tilde{\alpha}_N\), for each \(\mathcal{I}_N^m\), can be written as the following linear program

\[
\max_{\mathcal{I}_N^m, z(\mathcal{I}_N^m)} \frac{\eta_N}{\sum_{\mathcal{I}_N^m} \gamma_N(\mathcal{I}_N^m)} z(\mathcal{I}_N^m)
\]

s.t. \(z(\mathcal{I}_N^m) \geq -\left((\tilde{\alpha}_N)^T[\beta\tilde{M}_{N+1}(\mathcal{I}_N^m) - \tilde{M}_N]\right), z(\mathcal{I}_N^m) \geq 0 \forall \mathcal{I}_N^m\)

The dual of the above linear program is then

\[
\min_{\mathcal{I}_N^m, x(\mathcal{I}_N^m)} \psi_N(\mathcal{I}_N^m) \times 0
\]

s.t. \(0 \leq \psi_N(\mathcal{I}_N^m) \leq \frac{\gamma_N(\mathcal{I}_N^m)}{\eta_N} \forall \mathcal{I}_N^m, \sum_{\mathcal{I}_N^m} \psi_N(\mathcal{I}_N^m) = 1, \sum_{\mathcal{I}_N^m} \psi_N(\mathcal{I}_N^m)\beta M_{N+1}^j(\mathcal{I}_N^m) = M^j_N \forall j\)

By the partial markets assumption, there is a unique solution, namely \(\psi_N(\mathcal{I}_N^m) = \pi_N(\mathcal{I}_N^m)\) for each \(\mathcal{I}_N^m\), which is feasible for the above dual problem and the optimal objective function value of the dual is equal to 0. By strong duality, a feasible solution exists for the primal problem with the optimal objective function of the primal also equal to 0. It is easy to verify that \(\tilde{\alpha}_N = (0, \ldots, 0)\) is an optimal solution to the primal problem. Substituting \(\tilde{\alpha}_N\), we get \(H_N(\tilde{X}, \tilde{\alpha}_{N-1}, \mathcal{I}_N^m) = (\tilde{\alpha}_{N-1})^T\tilde{M}_N + CVaR_{(\eta_{N-1},N)}(X_N)\).
The theorem is therefore true for period $N$. Suppose it is true for periods $n + 1, \ldots, N$. Let

$$C_{n+1}(T_{n+1}) = \mathbb{E}_{\pi_{n+1}} \left[ \sum_{\tau = n+1}^{T} \beta^{\tau-(n+1)} CVaR_{(\eta_{\tau}, \tau)}^{(p)}(X_{\tau}) \right]$$

From equation (4), we have

$$H_n(\tilde{X}, \tilde{\alpha}_{n-1}, T_n^m) = \max_{\tilde{\alpha}_n} \left\{ [\tilde{\alpha}_{n-1} - \tilde{\alpha}_n]^T \tilde{M}_n + CVaR^{(p)}_{(\eta_n, n)}(X_n) ight.$$  

$$+ CVaR^{(m)}_{(\eta_n, n)} \left( \beta \tilde{\alpha}_n^T \tilde{M}_{n+1}(T_{n+1}^m) + \beta C_{n+1}(T_{n+1}^m) \right) \}$$  

$$= (\tilde{\alpha}_{n-1})^T \tilde{M}_n + CVaR^{(p)}_{(\eta_n, n)}(X_n)$$  

$$+ \max_{\tilde{\alpha}_n} \left\{ CVaR^{(m)}_{(\eta_n, n)} \left( (\tilde{\alpha}_n)^T [\beta \tilde{M}_{n+1}(T_{n+1}^m) - \tilde{M}_n] + \beta C_{n+1}(T_{n+1}^m) \right) \right\}$$

where we have used the translation invariance property of coherent risk measures to get the second equality. As before, writing the maximization over $\tilde{\alpha}_n$ as a linear program we get

$$\max_{\tilde{\alpha}_n, v, z(T_{n+1}^m)} v - \frac{1}{\eta_n} \sum_{T_{n+1}^m} \gamma_n(T_{n+1}^m)z(T_{n+1}^m)$$  

$$\text{s.t. } z(T_{n+1}^m) \geq v - \left( (\tilde{\alpha}_n)^T [\beta \tilde{M}_{n+1}(T_{n+1}^m) - \tilde{M}_n] + \beta C_{n+1}(T_{n+1}^m) \right), z(T_{n+1}^m) \geq 0 \ \forall T_{n+1}^m$$

The dual of the above linear program is then

$$\min_{\psi_n(T_{n+1}^m)} \sum_{T_{n+1}^m} \psi_n(T_{n+1}^m) \beta C_{n+1}(T_{n+1}^m)$$  

$$\text{s.t.}$$  

$$0 \leq \psi_n(T_{n+1}^m) \leq \frac{\gamma_n(T_{n+1}^m)}{\eta_n} \ \forall T_{n+1}^m, \ \sum_{T_{n+1}^m} \psi_n(T_{n+1}^m) = 1, \ \sum_{T_{n+1}^m} \psi_n(T_{n+1}^m) \beta M_{n+1}(j, T_{n+1}^m) = M_n(j) \ \forall j$$

By the partial markets assumption, there is a unique solution, namely $\psi_n(T_{n+1}^m) = \pi_n(T_{n+1}^m)$ for each $T_{n+1}^m$, which satisfy the set of linear equalities in the above minimization. By the conditions of the theorem, the risk-neutral probabilities also satisfy the inequalities in the above problem. Substituting this, we get

$$H_n(\tilde{X}, \tilde{\alpha}_{n-1}, T_n^m) = (\tilde{\alpha}_{n-1})^T \tilde{M}_n + CVaR^{(p)}_{(\eta_n, n)}(X_n) + \beta \mathbb{E}_{\pi_n} [C_{n+1}(T_{n+1}^m)]$$  

$$= (\tilde{\alpha}_{n-1})^T \tilde{M}_n + \mathbb{E}_{\pi_n} \left[ \sum_{\tau = n}^{N} \beta^{\tau-n} CVaR^{(p)}_{(\eta_{\tau}, \tau)}(X_{\tau}) \right]$$

□

**Proof of Lemma 1** We use induction to prove the lemma. Consider the situation where $n$ is such that $N_{L-1} \leq n < N_L$. Without loss of generality, we assume $N_L = N$. In period $N - 1$, it is optimal to commit all available output, $Q_{N-1} + y_{N-1} - R_{N-1}^{N_N}$, for sale because any left over output inventory does not earn
revenue in the subsequent period. Substituting the optimal commitment quantity in equation (8), we get
\[ V_{N-1}(e_{N-1}, Q_{N-1}, R_{N-1}^{NL}, I_{N-1}) = \beta F_{N-1}^{NL} \times (Q_{N-1} - R_{N-1}^{NL}) + U_{N-1}(e_{N-1}, I_{N-1}) \]
where
\[ U_{N-1}(e_{N-1}, I_{N-1}) = \max_{x_{N-1}, y_{N-1}} \left\{ \left( \beta F_{N-1}^{NL} - p_{N-1} \right) y_{N-1} - S_{N-1} x_{N-1} + \beta \mathbb{E}_{\pi_{N-1}} \left[ CVaR\{ S_N \times e_N \} \right] \right\} \]
\[ \text{s.t. } 0 \leq e_{N-1} = e_{N-1} + x_{N-1} - y_{N-1}, 0 \leq y_{N-1} \leq C, 0 \leq x_{N-1} \leq K \]

Because CVaR is a coherent risk measure, it satisfies the positive homogeneity condition. Therefore, we have \( \beta \mathbb{E}_{\pi_{N-1}} \left[ CVaR_{\pi_{N-1}}(S_N \times e_N) \right] = \beta \mathbb{E}_{\pi_{N-1}} \left[ CVaR_{\pi_{N-1}}(S_N) \right] \times e_N \). Notice therefore that the objective function in the maximization problem that yields \( U_{N-1}(e_{N-1}, I_{N-1}) \) is linear in the decision variables, as are the constraints. Thus, \( U_{N-1}(\cdot) \) is the solution of a linear program where \( e_{N-1} \) is on the right hand side of the constraints, and \( U_{N-1}(\cdot) \) is piecewise linear and concave in \( e_{N-1} \) for each realization of \( I_{N-1} \).

Also notice that \( V_{N-1} \) is separable in \( Q_{N-1} \) and \( e_{N-1} \) and furthermore, linear in \( Q_{N-1} \) with the coefficient of \( Q_{N-1} \) dependent only on \( T_N^{N-1} \).

Because the revenue from selling a unit of output depends only on market uncertainty, we can use induction arguments similar to those in Devalkar et al. (2011) and show that \( V_n(\cdot) \) is separable in \( Q_n \) and that the optimal commitment policy and marginal value of output inventory is as given in the Lemma. \( \square \)

**Proof of Theorem 2** We prove the theorem by induction. The theorem is true for \( n = N \), with \( \Theta_{N}^{(k)} = S_N \) for all \( k \geq 1 \). Let \( T_{n+1}^{p} \) and \( T_{n+1}^{p} \) be such that \( S_{n+1}(T_{n+1}^{m}, T_{n+1}^{p}(i)) > S_{n+1}(T_{n+1}^{m}, T_{n+1}^{p}(j)) \).

Using the shorthand notation \( g_n(\cdot, i) \) to denote \( g_n(\cdot, T_{n+1}^{p}(i)) \), we can verify that \( U_{N-1}(e_{N-1}, i) \leq U_{N-1}(e_{N-1}, j) \) and \( \Theta_n^{(k)}(i) \geq \Theta_n^{(k)}(j) \) for all \( k \), for all \( e_{N-1} \in [(k-1)D, kD] \). That is, in period \( N-1 \), the value function is non-increasing, and the marginal value of input inventory is non-decreasing, in the realized spot price. Suppose the theorem and the above properties are true for periods \( n+1, \ldots, N-1 \).

In period \( n+1 \), we have \( U_{n+1}(e_{n+1}, I_{n+1}) = \Theta_{n+1}(e_{n+1} + \Lambda_{n+1}) \) for \( e_{n+1} \in [(k-1)D, kD] \), with \( \Delta_{n+1} \geq \Theta_{n+1}^{(k+1)} \). We also have \( U_{n+1}(e_{n+1}, i) \leq U_{n+1}(e_{n+1}, j) \) and \( \Theta_{n+1}(i) \geq \Theta_{n+1}(j) \) for all \( k \) for all \( e_{n+1} \in [(k-1)D, kD] \), when \( S_{n+1}(i) > S_{n+1}(j) \) for a given \( T_{n+1}^{m} \), by the induction hypothesis.

In period \( n \), we have
\[ U_{n}(e_{n}, I_{n}) = \max_{0 \leq x \leq K} \left\{ -S_n x + \max_{0 \leq y \leq \min(\{C, e_{n} + x\})} \left\{ \left( \Delta_{n} - p_n \right) y + \beta \mathbb{E}_{\pi_{n}} \left[ CVaR\{ S_n \times e_n \} \right] \right\} \right\} \]
\[ = \max_{0 \leq x \leq K} \left\{ \max_{0 \leq y \leq \min(\{C, e_{n} + x\})} \left\{ \left( \Delta_{n} - p_n \right) y + \Psi_{n}^{(k)}(x - y) \right\} \right\} - S_n x \]
for \( e_{n} + x - y \in [(k-1)D, kD] \) for each \( k \geq 1 \), where \( \Psi_{n}^{(k)} = \beta \mathbb{E}_{\pi_{n}} \left[ CVaR\{ S_n \times e_n \} \right] \cdot \Lambda_{n+1}^{(k)} \). The second equality above follows from the fact that \( U_{n+1} \) and \( \Theta_{n+1}^{(k)} \) are non-increasing and non-decreasing, respectively, in \( S_{n+1} \).
Denote by $\mathcal{L}_n(w_n, \mathcal{I}_n^m) = \max_{0 \leq y_n \leq \min(c, w_n)} \left\{ (\Delta_n^m - p_n) \times y + \gamma_n^{(k)} \times (w_n - y) + \psi_n^{(k)} \right\}$, when $w_n - y \in [(k-1)D, kD)$. By concavity of $U_{n+1}$ in $e_{n+1}$, we have $\gamma_n^{(k)}$ non-increasing in $k$. As a result, for $w_n \in [(s - 1)D, sD)$, we can determine the optimal processing quantity as

$$y_n^* = \begin{cases} 0 & \text{if } \gamma_n^{(s)} > \Delta_n^m - p_n \\ w_n - \hat{r}_nD & \text{if } \gamma_n^{(s-a)} > \Delta_n^m - p_n \geq \gamma_n^{(s)} \\ C & \text{if } \Delta_n^m - p_n \geq \gamma_n^{(s-a)} \end{cases}$$

where $\hat{r}_n = \max\{r \in \mathbb{Z}^+ \text{ s.t. } \gamma_n^{(r)} > \Delta_n^m - p_n\}$. Substituting $y_n^*$, we can write

$$\mathcal{L}_n(w_n, \mathcal{I}_n^m) = \begin{cases} \gamma_n^{(s)} w_n + \psi_n^{(s)} & \text{if } \gamma_n^{(s)} > \Delta_n^m - p_n \\ (\Delta_n^m - p_n) w_n + \left( \gamma_n^{(s-a)} - (\Delta_n^m - p_n) \right) \hat{r}_nD + \psi_n^{(s)} & \text{if } \gamma_n^{(s-a)} > \Delta_n^m - p_n \geq \gamma_n^{(s)} \\ \gamma_n^{(s-a)} w_n + ((\Delta_n^m - p_n) - \gamma_n^{(s-a)}) C + \psi_n^{(s-a)} & \text{if } \Delta_n^m - p_n \geq \gamma_n^{(s-a)} \end{cases}$$

for each of the corresponding cases. More compactly, we can write $\mathcal{L}_n(w_n, \mathcal{I}_n^m) = \Omega_n^{(s)} w_n + \Phi_n^{(s)}$ where $\Omega_n^{(s)} = \max \left\{ \gamma_n^{(s)}, \min \left\{ \Delta_n^m - p_n, \gamma_n^{(s-a)} \right\} \right\}$ and $\Phi_n^{(s)}$ is the constant term that does not depend on $w_n$, when $w_n \in [(s - 1)D, sD)$. Notice that $\mathcal{L}_n$ is piecewise linear and concave in $w_n$ with breakpoints at integral multiples of $D$ and is $\mathcal{I}_n^m$-measurable.

We now have $U_n(e_n, \mathcal{I}_n) = \max_{0 \leq w_n \leq e_n + K} \left\{ \Omega_n^{(s)} w_n + \Phi_n^{(s)} - S_n \times (w_n - e_n) \right\}$, when $w_n \in [(s - 1)D, sD)$. From the induction assumption about concavity of $U_{n+1}$, we have $\Omega_n^{(s)}$ non-increasing in $s$. As a result, for $e_n \in [(k-1)D, kD)$ where $k$ is a positive integer, we can determine the optimal procurement quantity $x_n^* = w_n^* - e_n$ as

$$x_n^* = \begin{cases} K & \text{if } \Omega_n^{(k+b)} \geq S_n \\ \hat{s}_n D - e_n & \text{if } \Omega_n^{(k)} \geq S_n > \Omega_n^{(k+b)} \\ 0 & \text{if } S_n > \Omega_n^{(k)} \end{cases}$$

where $\hat{s}_n = \max\{s \in \mathbb{Z}^+ \text{ s.t. } \Omega_n^{(s)} > S_n\}$. Substituting the optimal $w_n^*$ values from equation (20), we get

$$U_n(e_n, \mathcal{I}_n) = \begin{cases} \Omega_n^{(k+b)} e_n + (\Omega_n^{(k+b)} - S_n) K + \Phi_n^{(k+b)} & \text{if } \Omega_n^{(k+b)} \geq S_n \\ S_n e_n + (\Omega_n^{(k)} - S_n) \hat{s}_n D + \Phi_n^{(k)} & \text{if } \Omega_n^{(k)} \geq S_n > \Omega_n^{(k+b)} \\ \Omega_n^{(k)} e_n + \Phi_n^{(k)} & \text{if } S_n > \Omega_n^{(k)} \end{cases}$$

for each of the three cases. Thus, we can write $U_n(e_n, \mathcal{I}_n) = \max \left\{ \Omega_n^{(k+b)}, \min \{S_n, \Omega_n^{(k)}\} \right\} e_n + \Lambda_n(k)$ where $\Lambda_n(k)$ represents the constant terms not dependent on $e_n$, when $e_n \in [(k-1)D, kD)$ for $k \geq 1$.

To finish the proof, consider three realizations of $S_n$ such that $S_n(1) > \Omega_n^{(k)} \geq S_n(2) > \Omega_n^{(k+b)} \geq S_n(3)$. We have

$$U_n(e_n, 3) = \Omega_n^{(k+b)} e_n + (\Omega_n^{(k+b)} - S_n(3)) K + \Phi_n^{(k+b)}$$

$$\geq \Omega_n^{(k+b)} e_n + (\Omega_n^{(k+b)} - S_n(3)) (\hat{s}_n D - e_n) + \Phi_n^{(k+b)} = \Omega_n^{(k+b)} \hat{s}_n D - S_n(3) (\hat{s}_n D - e_n) + \Phi_n^{(k+b)}$$

$$\geq \Omega_n^{(k+b)} \hat{s}_n D - S_n(2) (\hat{s}_n D - e_n) + \Phi_n^{(k+b)}$$

$$\geq \Omega_n^{(k)} \hat{s}_n D - S_n(2) (\hat{s}_n D - e_n) + \Phi_n^{(k)} = U_n(e_n, 2)$$

$$\geq \Omega_n^{(k)} e_n - S_n(2) (e_n - e_n) + \Phi_n^{(k)}$$
\[ \Omega_n^{(k)} e_n + \Phi_n^{(k)} = U_n(e_n, 1) \]

The first inequality above follows from the fact that \( K \geq \hat{s}_n D - e_n \), the second because \( S_n(2) > S_n(3) \), the third by concavity of \( L_n \), the fourth because \( \hat{s}_n D \geq e_n \), and the last by concavity of \( L_n \). Thus, \( U_n(e_n, \mathcal{I}_n) \) is decreasing in \( S_n \), while from equation (21) we see that \( \Theta_n^{(k)} \) is increasing in \( S_n \) for a given realization of \( \mathcal{I}_n \). Therefore by induction, the theorem is true for all \( n \). \( \square \)

**Proof of Theorem 3** Denoting the various parameters evaluated for the vector \( \tilde{\eta} \) with a ‘, we have \( \Theta_n^{(k)} = \Theta_N^{(k)} = S_N \) for all \( k \geq 1 \). Further, \( CV a R^{(p)}_{(\eta_N^{(k)}, N)} \left( \Theta_n^{(k)} \right) = CV a R^{(p)}_{(\eta_N, N)} \left( \Theta_N^{(k)} \right) = CV a R^{(p)}_{(\eta_N, N)}(S_N) \) for all \( \mathcal{I}_N \), since \( \eta_N^{(k)} = \eta_N \). Thus, the theorem is true for \( n = N \).

Suppose the theorem is true for \( n + 1, \ldots, N \). We have
\[
\Theta_n^{(k)} \geq \Theta_{n+1}^{(k)} \Rightarrow -\Theta_n^{(k)} \leq -\Theta_{n+1}^{(k)} \\
\Rightarrow CV a R^{(p)}_{(\eta_n^{(k)}, n+1)} \left( -\Theta_n^{(k)} \right) \leq CV a R^{(p)}_{(\eta_n^{(k)}, n+1)} \left( -\Theta_{n+1}^{(k)} \right) \\
\Rightarrow -CV a R^{(p)}_{(\eta_n^{(k)}, n+1)} \left( -\Theta_n^{(k)} \right) \geq -CV a R^{(p)}_{(\eta_n^{(k)}, n+1)} \left( -\Theta_{n+1}^{(k)} \right) \\
\Rightarrow \beta \mathbb{E}_n \left[ -CV a R^{(p)}_{(\eta_n^{(k)}, n+1)} \left( -\Theta_n^{(k)} \right) \right] \geq \beta \mathbb{E}_n \left[ -CV a R^{(p)}_{(\eta_n^{(k)}, n+1)} \left( -\Theta_{n+1}^{(k)} \right) \right] \Rightarrow \Upsilon_n^{(k)} \geq \Upsilon_{n+1}^{(k)}
\]

where the inequality (22a) follows from the fact that \( CV a R \) is a coherent risk measure and therefore monotonic, and \( \eta_n^{(k)} \leq \eta_{n+1}^{(k)} \). From equations (14)–(13), we have \( \Omega_n^{(k)} \geq \Omega_n^{(k)} \) and as a result, \( \Theta_n^{(k)} \geq \Theta_n^{(k)} \) for all \( k \geq 1 \).

From Corollary 1, we have \( x_n^{*} \geq x_n^{*} \) for all \( \mathcal{I}_n \) when \( e_n = e_n \), because \( \Omega_n^{(k)} \geq \Omega_n^{(k)} \) for all \( k \). \( \square \)