Essays on Multi-Item Auctions

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Adithya Patil (ID: 51650002)

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Abstract

Essays on Multi-Item Auctions

Author: Adithya Patil (ID: 51650002)

Chair of the Dissertation Committee: Milind G. Sohoni

Members of the Dissertation Committee: Sumit M. Kunnumkal, Nishant Ravi, Jayashankar Swaminathan

Multi-Item auctions are of interest to companies that run online auctions. In this dissertation, I examine auctions involving multiple items in three contexts.

The first context involves revenue improvement in the simultaneous auctions of multiple items when the number of bidders and the number of bids for each item are known. In this work, the lever for revenue improvement I examine is item-bundling. Specifically, I study the problem of bundling items together in a manner that improves the seller's revenue prior to auctioning them off in simultaneous second-price auctions. I propose an auction format, called the Pairwise Bundle Auction (PBA), that elicits truthful bids from bidders for the items on sale. I provide a mathematical formulation that computes the revenue-maximizing bundling of items in response to the bids submitted. My work on identifying a revenue-maximizing bundling of items is of use to companies that run online auctions as a core part of revenue-generation. Examples include companies such as Google or Facebook that run auctions to sell advertisement slots.

The second context involves minimizing the cost of uncertainty in the simultaneous auctions of multiple items when the number of bidders and the number of bids for each item are uncertain. In this work, the lever for revenue improvement

I examine is limiting item supply. When a set of items is put out for auction by a seller, the uncertainty in the bidders' participation decisions can result in adverse outcomes for the seller. I refer to this as the "cost of uncertainty". Therefore, the seller would want to restrict the set of items put up for sale to minimize the cost of uncertainty. I formulate the problem of identifying an optimal subset of items to put up for simultaneous auction out of a master set of items. This optimal subset minimizes the maximum regret arising from the uncertainty in the bidders' participation decisions. Our results focus on the computational complexity of this problem. My work on identifying a maximum regret minimizing subset of items to put up for sale is of use to companies that auction off items on online platforms (such as eBay) where participation decisions are uncertain.

The third context involves maximizing the total welfare from item allocations to bidders when allocative externalities are involved. Under the externality model I consider, the value of an item to a bidder depends on the allocation of the other items to the other bidders. I identify a class of valuation functions called the Pairwise Additive Negative Value Externalities (PANE) with interesting properties. I show that the PANE class of valuation functions correspond to anonymous and simple pricing structures that support a social-welfare-maximizing allocation of items to bidders. Like the first context, my work on identifying this class of valuation functions is of use to companies that run online auctions as a core component of revenue generation such as Google or Facebook.

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Dedication

I dedicate this thesis to my parents.

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Chapter 1

Introduction

There has been a massive growth of research in multi-item auction design in recent years. Many of the problems studied are of importance to the business of online auctions. Examples of online include those of Facebook's advertisement slot auctions and those of eBay's auctions whose transactions generate revenues to the tune of billions of dollars.

Auction design primarily addresses two objectives of an auctioneer: (*i*) maximizing the auctioneer's revenue, and (*ii*) maximizing the social welfare of the bidders from the allocation of items to the bidders.

In this thesis, I address important questions around auction designs for maximizing the auctioneer's revenue and for maximizing the social welfare of the bidders in contexts that have not been explored in literature. In §1.1, I discuss in detail the broad areas of research in multi-item auction design that my thesis has focused on. I explain the importance of my work for business and for research and discuss how my thesis helps answer some pressing questions in multi-item auction design. In §1.2, I discuss the specifics of my contribution to the literature on revenue maximization and social-welfare maximization. I summarize the contributions of my papers detailed in Chapters 3, 4, and 5, and conclude my work with a brief discussion on the future research for which my thesis has laid out a path.

1.1 Overview of Research Areas

1.1.1 Revenue Improvement for Multi-Item Auctions

Finding a revenue-maximizing sales procedure for the sale of multiple items to multiple bidders is an important goal in many business contexts. For example, governments across the world auction spectrum and natural resources, and advertising platforms auction off advertisement slots. In both these instances, there are several items and several buyers interested in acquiring one or more of the items. In all these cases, identifying a sales procedure (or a mechanism) that can maximize the seller's revenue is an important problem. Mechanism design is an approach taken to design a set of rules meant to be followed by agents to achieve a specific goal. Thus, designing a revenue-maximizing mechanism would entail a study of incentives with approaches to aligning the incentives with the objectives of the seller. Auction design is an important subclass of mechanism design. The aim of auction design is to construct a set of rules that buyers are expected to compete under in order to acquire one or more of the items on sale. In this regard, identifying an auction design that maximizes the revenue of the seller is of business importance.

An open problem in multi-item auction design is that of designing a revenue-maximizing mechanism for multi-item auctions. It is an open problem because, thus far, there has been no characterization of a revenue-maximizing mechanism even for a problem instance with two items and two bidders. The classic work by Myerson (1981) characterizes the optimal revenue-maximizing auction for a single-item setting when the distribution function of the bidders' valuations are known. Nisan et. al. (2007) presents a characterization of the result of Myerson (1981) to multiple copies of a single item. In addition to this, the works of Avery et. al. (2000) and Armstrong (2000) provide a characterization of the multi-item revenue-maximizing auction to some special cases of the two-item two-bidder setting. The work of Daskalakis (2015) provides some insight into the characterization of the optimal mechanism. It shows how some features of the optimal mechanism include the following: (*i*) item bundling, (*ii*) randomization, and (*iii*) complex menus.

Some of the above features, namely randomization and generating complex menus, are often not possible in practice. Randomization involves assigning a winner based on a lottery the probabilities of whose outcomes are determined in a certain manner. Offering the bidders complex menus is not possible in practice, generally speaking, because the number of options included in the menu could be uncountable. As a result, even if the optimal auction were to be somehow fully characterized, its implementation in practice would be difficult. Thus, the problem of characterizing the revenue-maximizing mechanism for the sale of multiple items is complex. Owing to the complexity of designing the revenue-maximizing multi-item auction, literature has often focused on examining suboptimal but simpler mechanisms that would yield high revenue over existing mechanisms used in practice for the sale of multiple items.

Bundling is one approach towards revenue improvement. Bundling does not depend on randomization, does not rely on the need for complex menus, and is used in practice as a means to improve revenue in sales. There are several pieces of work in literature that study bundling as a means of improving revenue. Such work includes those of Adams et. al. (1976), Guiltinan (1987), Hanson et. al. (1990), Hitt et. al. (2005), Wu et. al. (2008), Chu et. al. (2011), Palfrey (1983), Sandholm et. al. (2004), Jehiel et. al. (2007). As a result, implementing bundling as a lever to improve revenue is not difficult to implement in practice.

Another avenue for revenue improvement is that of offering a subset of the set of items on sale instead of all items on sale. This is known in literature as assortment optimization. The set of items on offer affect the buyers' purchasing decisions, and optimizing this offer set to maximize revenue is often a goal of a seller. Like bundling, this avenue for revenue improvement does not depend on randomization or complex menus. It can be implemented easily in practice. Literature on assortment optimization focus on settings where there is a single buyer with the aim of maximizing expected revenue. In the case of multiple buyers, as is the case with competitive bidding, literature has not studied the problem of optimizing the offer set. In my thesis, I examine the question of identifying an offer set under a simultaneous auction setting.

In conclusion, I discuss the two avenues of revenue improvement in the context of multi-item auctions in Chapters 3 and 4 of this thesis. Chapter 3 discusses the problem of identifying an optimal bundling in a VCG auction setting. Chapter 4 discusses the problem of optimizing an offer set in a second-price simultaneous auction setting.

1.1.2 Social-Welfare-Maximizing Allocation for Multi-Item Auctions

In many business contexts, maximizing the social welfare of the buyers arising from an allocation is of importance. This objective is often in conflict with the objective of revenue maximization. For example, it is important for advertising platforms hosting advertisements to try and ensure that bidders who value an advertisement slot the most are allotted the advertisement slot. This way, the auctioneer can ensure that the advertisers continue to use the services of the platform in the long term instead of considering a competitor. This is because revenue gains are often brought about by extracting surplus from the advertisers, and frequently favoring some advertisers as part of a revenue-maximizing auction design may hurt the long term revenue of a platform. For this reason, the objective maximizing social welfare is an important goal of the auctioneer.

For this reason, literature has paid a lot of attention to business contexts where the goal of auction design is to maximize the social welfare of the bidders arising from the allocation of items. An important class of auction mechanisms that always result in allocations that maximize the total social welfare of the bidders is the Vickrey-Clarke-Groves auction (VCG). While the VCG can be used to maximize social welfare, it is a direct mechanism, and would require complete information revelation to the auctioneer. On the other hand, indirect mechanisms are preferred in practice as they don't require much information revelation, and also because the computational burden on the bidders is low. Thus, indirect mechanisms that result in a social-welfare-maximizing outcome is of importance for study. Literature has examined a number of settings where the social-welfare-maximizing allocation is obtained through indirect mechanisms. Such work includes those of Cramton (1998), de Vries et. al. (2007), Demange et. al. (1986), Bikhchandani et. al. (2002), Parkes (2001) and Candogan et. al. (2015) and the references therein.

The pieces of work in the above listed references (and the references therein) examine the problem of designing indirect mechanisms for multi-item sales for a variety of classes of the bidders' valuation functions. However, to the best of my knowledge, none of them have studied the problem of designing indirect mechanisms when the bidders' valuation functions include allocative externalities, i.e., settings where a bidder's valuation of an allocation depends not just on the items he has been allocated, but also on the allocation of items to the other bidders.

I attempt to bridge this gap in literature by examining the question of designing indirect mechanisms for social-welfare-maximization in a setting where the bidders' valuation functions also consider allocative externalities. This study is detailed in Chapter 5 of this thesis.

1.2 Contributions to Literature

In this section, I summarize the contributions the chapters of my thesis makes to literature in the area of designing auctions for revenue-enhancement in multi-item auctions and the area of designing auction mechanisms for social-welfare maximizing allocation of items. My contributions to literature are titled as follows:

1. Optimal Bundling for Truthful Auctions

2. Optimizing Offer Sets for Multi-Item Simultaneous Auctions

3. Auction Mechanisms for Social-Welfare-Maximizing Allocations with Pairwise-Additive Negative Value Externalities

1.2.1 Optimal Bundling for Truthful Auctions

In this chapter, we examine the question of computing the optimal bundling of items if these bundles were to be sold using a VCG auction. In this regard, I propose a class of truthful auctions called the PBA. The auctioneer can use the PBA to compute the optimal bundling of items after the bidders reveal their valuations of the items to the auctioneer. I propose a binary integer programming formulation to compute the optimal bundling. This binary integer program is obtained after a series of simplifications to a more complex binary cubic optimization problem. I then present results from numerical runs and discuss insights from them. Apart from these results, I present a class of linearly-constrained binary quadratic optimization problems whose relaxations would still provide integral solutions at optimality.

1.2.2 Optimizing Offer Sets for Multi-Item Simultaneous Auctions

I consider the auctioneer's problem of optimizing the set of items to put out for auction when the auction is a multi-item simultaneous second-price auction. Here, I highlight the fact that the uncertainty in the bidders' participation decisions (for a given set of items put out for auction) can impact the auctioneer's revenue from the auction. I illustrate how putting up a certain set of items can give the auctioneer a very high best-case expected revenue or a very low worst-case revenue. Therefore, it is necessary that the auctioneer choose a subset of items keeping in mind this uncertainty. In this regard, I use a minimax regret criterion to identify an optimal subset of items to put up for sale. To the best of our knowledge, there is no prior work of this form in literature.

1.2.3 Auction Mechanisms for Social-Welfare-Maximizing Allocations with Pairwise-Additive Negative Value Externalities

I examine a class of valuation functions that captures negative externalities in allocations. The social welfare maximization problem under this class of valuation functions is a binary quadratic program with binary linear constraints. I show that the optimal solution to this optimization problem is binary even after the binary constraints are relaxed. I also show that the relaxed program exhibits strong duality. With these results, I show that under these valuation functions, a social-welfare-maximizing allocation of items to bidders can be brought about using simple and anonymous item prices[1]. Finally, I propose two auction formats, a direct mechanism and an indirect mechanism, that terminate at a social-welfare-maximizing allocation.

I discuss, in detail, the importance of identifying valuation functions for which simple and anonymous items exist in Chapter 2. Chapter 2, therefore, may be a prerequisite for readers not familiar with auction theory literature in multi-item auction settings.

Chapter 2

Auction Design and Linear Programming

2.1 Linear Programming Theory

The following is a standard representation of a linear programming problem. I call this problem P.

$$\textbf{P:} \qquad \max \mathbf{c}^T\mathbf{x} \tag{2.1}$$

$$\text{subject to} \qquad \mathbf{A}\mathbf{x} \leqslant \mathbf{b} \tag{2.2}$$

$$\mathbf{x} \geqslant \mathbf{0} \tag{2.3}$$

Here, $\mathbf{A}$ is an $m \times n$ matrix with real values, $\mathbf{c}$ is an $n-$dimensional vector, $\mathbf{b}$ is an $m-$dimensional vector, and $\mathbf{x}$ is an $n-$dimensional vector. Problem P, the primal problem, is to compute a solution $\mathbf{x}$ that maximizes $\mathbf{c}^T\mathbf{x}$ subject to the constraints. The dual problem D of problem P is the following problem.

$$\textbf{D:} \qquad \min \mathbf{p}^T\mathbf{b} \tag{2.4}$$

$$\text{subject to} \qquad \mathbf{A}^T\mathbf{p} \geqslant \mathbf{c} \tag{2.5}$$

$$\mathbf{p} \geqslant \mathbf{0} \tag{2.6}$$

Here, $\mathbf{p}$ is an $m-$dimensional vector. Problem D, the dual problem, is to compute a solution $\mathbf{p}$ that minimizes $\mathbf{p}^T\mathbf{b}$ subject to the constraints.

Let $\mathbf{x}^*$ be the optimal solution to P and let $\mathbf{p}^*$ be the optimal solution to D. According to the weak duality theorem, $\mathbf{c}^T\mathbf{x}^* \leqslant \mathbf{p}^{*T}\mathbf{b}$. In other words, the dual objective function value at optimality is an upper bound to the primal objective function value at optimality. According to the strong duality theorem, the primal and the dual optimal objective values are optimal if and only if the objective value of the primal equals the objective value of the dual. Thus, if $\mathbf{x}$ and $\mathbf{p}$ optimize P and D respectively, then $\mathbf{c}^T\mathbf{x} = \mathbf{p}^T\mathbf{b}$.

The strong duality theorem can also be expressed in terms of conditions known as the complementary slackness conditions. The primal complementary slackness conditions can be stated as follows:

$$\mathbf{x}^T(\mathbf{A}^T\mathbf{p} - \mathbf{c}) = 0 \tag{2.7}$$

or, in other words, $x_i > 0 \iff \mathbf{A}_i^T\mathbf{p} = c_i$ where x_i is the i^{th} component of vector $\mathbf{x}$, $\mathbf{A}_i$ is the i^{th} column of matrix $\mathbf{A}$, and c_i is the i^{th} component of vector $\mathbf{c}$. The dual complementary slackness conditions can be stated as follows:

$$\mathbf{p}^T(\mathbf{A}\mathbf{x} - \mathbf{b}) = 0 \tag{2.8}$$

or, in other words, $p_j > 0 \iff \mathbf{a}_j\mathbf{x} = b_j$ where p_j is the j^{th} component of vector $\mathbf{p}$, $\mathbf{a}_j$ is the j^{th} row of matrix $\mathbf{A}$, and b_j is the j^{th} component of vector $\mathbf{b}$. Thus, if $\mathbf{x}$ and $\mathbf{p}$ optimize P and D respectively, then $\mathbf{x}$ and $\mathbf{p}$ satisfy the primal and dual complementary slackness conditions, or simply, the complementary slackness conditions.

2.2 Winner Determination in Simultaneous Auctions with Additive Valuations

In the context of auctions with one or more indivisible items, where an item can only go to one bidder, a goal of the auctioneer is to decide the winners of the auctions using a metric. One metric could be to maximize the welfare of the bidders resulting from the allocation. Let L be the set of items, and let B be the set of bidders. Let bidder i value item a at an amount $u_i^a > 0$. Since the auctioneer's goal is to maximize the welfare of the bidders resulting from the allocation, the auctioneer solves the following integer program IP.

$$\textbf{IP:} \qquad \max \sum_{i \in B} \sum_{a \in L} u_i^a x_i^a \tag{2.9}$$

$$\text{subject to} \qquad \sum_{i \in B} x_i^a = 1 \quad \forall\ a \in L \tag{2.10}$$

$$x_i^a \in \{0, 1\} \quad \forall i \in B \quad \forall\ a \in L \tag{2.11}$$

The objective function maximizes the total utility of the bidders from an allocation of items. Here, $x_i^a = 1$ if bidder i is assigned item a and $x_i^a = 0$ if bidder i is not assigned item a. The first constraint models the fact that an item a can only be assigned to one bidder. The second constraint constrains the decision variables to binary values. The solution to IP is as follows: If $i_a = \arg\max_{j \in B} u_j^a$, then $x_{i_a}^a = 1$ for all $a \in L$. Else, $x_{i_a}^a = 0$. In other words, the bidder who has the highest value for item a, $a \in L$, wins item a.

Consider the following linear program constructed from relaxing the binary integer constraints in IP. I call this problem LP.

$$\textbf{LP:} \qquad \max \sum_{i \in B} \sum_{a \in L} u_i^a x_i^a \tag{2.12}$$

$$\text{subject to} \qquad \sum_{i \in B} x_i^a = 1 \quad \forall\ a \in L \quad \cdots (p^a) \tag{2.13}$$

$$x_i^a \leqslant 1 \quad \forall i \in B \quad \forall\ a \in L \quad \cdots (\rho_i^a) \tag{2.14}$$

$$x_i^a \geqslant 0 \quad \forall i \in B \quad \forall\, a \in L \tag{2.15}$$

where p^a and ρ_i^a are a Lagrangean multipliers associated with the respective constraints. Because the entries in the right-hand-side vector of LP are integers, and because the constraint matrix is Totally Unimodular, solving LP will yield a solution x_i^{a*} for all $i \in B, a \in L$ such that x_i^{a*} is either 0 or 1. The dual of LP can be constructed as follows. I call this problem DP.

$$\textbf{DP:} \qquad \min \sum_{a \in L} p^a + \sum_{i \in B} \sum_{a \in L} \rho_i^a \tag{2.16}$$

$$\text{subject to} \qquad u_i^a - p^a \leqslant \rho_i^a \quad \forall i \in B, a \in L \quad \cdots (x_i^a) \tag{2.17}$$

$$\rho_i^a \geqslant 0 \quad \forall i \in B, a \in L \tag{2.18}$$

Since the problem LP is degenerate, the dual has multiple optimal solutions. Two such optimal solutions are of interest from the perspective of auction theory which we shall discuss below: Firstly, note that at optimality of DP, we have that $\rho_i^a = \max\{u_i^a - p^a, 0\}$. The term ρ_i^a can be regarded as the surplus of bidder i w.r.t a. As I shall show, if $\rho_i^a \geqslant 0$, then bidder i is assigned item a. If $\rho_i^a \leqslant 0$, then bidder i is not assigned item a.

Case 1 (Outcome corresponds to a first-price auction). If bidder i reported his true valuation for item a, i.e., u_i^a, as a bid, then a possible solution to the dual variable p^a for all $a \in L$ could be

$$p^{a*} = \begin{cases} u_{i_a}^a \text{ if } i_a = \arg\max_{j \in B} u_j^a \\ 0 \text{ otherwise} \end{cases}$$

with $\rho_i^a = 0$ for all $i \in B, a \in L$. Thus, the price of item a, i.e., p^{a*} is equal to the highest bid for item a. Item a is assigned to the highest bidder and he pays his bid for item a. This corresponds to the outcome of a first price auction.

Case 2 (Outcome corresponds to a second-price auction). If bidder i reported his true valuation for item a, i.e., u_i^a, as a bid, then a possible solution to the dual variable p^a for all $a \in L$ could be

$$p^{a*} = \begin{cases} u^a_{k_a} \text{ if } k_a = \arg \text{second-highest}_{j \in B} u^a_j \\ 0 \text{ otherwise} \end{cases}$$

with $\rho^a_i > 0$ if $i_a = \arg\max_{j \in B} u^a_j$ and $\rho^a_i = 0$ otherwise for all $a \in L$. The price of item a, i.e., p^{a*} is equal to the second-highest value for item a. Item a is assigned to the bidder with the highest value for item a, and this bidder pays the second-highest valuation for item a as the price. This corresponds to the outcome of a second price auction.

The second-price auction interpretation is particularly interesting. If $\rho^a_i > 0$, it implies, from the complementary slackness conditions, that $x^a_i = 1$, and if $\rho^a_i = 0$, it implies, from the complementary slackness conditions, that $x^a_i = 0$. Thus, the bidder i who wins item a, i.e., $x^a_i = 1$, then his surplus $\rho^a_i = u^a_i - p^a$ is positive. If bidder i does not win item a, i.e., $x^a_i = 0$, then his surplus is $\max\{u^a_i - p^a, 0\} = 0$ since if $x^a_i = 0 \iff u^a_i - p^a \leqslant 0$ by the complementary conditions.

Thus, from the discussion above, the complementary slackness conditions represent the vector of item prices and allocations that can be regarded as outcomes of simultaneous second-price auctions. They are also known as the market-clearing conditions.

The prices $p^a, a \in L$ are known as equilibrium prices since they support feasible social-welfare maximizing allocations without dividing the items. In this setting, such prices exist because LP has integral optimal solutions. If, hypothetically-speaking, LP did not have integral solutions, then the prices $p^a, a \in L$ that support feasible social-welfare maximizing allocations without dividing the items would not exist. The fact that the bidders' valuations for multiple items are additive contributed to the existence of such prices. As we shall see next, if the bidders' valuations for multiple items were non-additive, prices of the form $p^a, a \in L$ would not exist, i.e., prices at the level of an item would not exist.

Note. In auction theory literature, prices that are set at the item level and are not dependent on the identity of the bidder are called simple and anonymous prices respectively.

2.3 Winner Determination when Valuations are Non-Additive

Let Ω be the power set of the item set L. Let the value of a bundle S of items to bidder i be $v_i(S)$. Let $x_i(S) = 1$ if bidder i receives item set S, and let $x_i(S) = 0$ if bidder i does not receive item set S. The allocation that results in maximizing the total utility of the bidders is the obtained by solving the following binary integer program I call NAIP.

$$\textbf{NAIP:} \qquad \max \sum_{S \in \Omega} \sum_{i \in B} v_i(S) x_i(S) \tag{2.19}$$

$$\text{subject to} \qquad \sum_{S \in \Omega} x_i(S) \leqslant 1 \quad \forall\ i \in B \tag{2.20}$$

$$\sum_{i \in B} \sum_{S \in \Omega : a \in S} x_i(S) \leqslant 1 \quad \forall\ a \in L \tag{2.21}$$

$$x_i(S) \in \{0, 1\} \quad \forall\ S \in \Omega \quad \forall\ i \in B \tag{2.22}$$

The first constraint ensures that bidder i only gets one bundle of items from Ω. The second constraint ensures that an item a is only present in one of the bundles that is allocated to the bidders. The third constraint ensures that the decision variables involved are binary-valued.

I now discuss how modifications to the problem NAIP can provide insight into the nature of the equilibrium pricing structures for non-additive bundle valuations. This analysis was presented originally in Bikhchandani et. al. (2002). Here, I present the insights of Bikhchandani et. al. (2002) without going into the details.

2.3.1 First-order Linear Programming Formulation

The first-order linear programming formulation corresponding to NAIP is constructed by simply relaxing the integrality constraints on the variables $x_i(S), i \in B, S \in \Omega$. I call the following linear program FO-NALP.

$$\textbf{FO-NALP:} \qquad \max \sum_{S \in \Omega} \sum_{i \in S} v_i(S) x_i(S) \tag{2.23}$$

$$\text{subject to} \qquad \sum_{S \in \Omega} x_i(S) \leqslant 1 \quad \forall\, i \in B \quad \cdots \quad (q_i) \tag{2.24}$$

$$\sum_{S \in \Omega : a \in S} x_i(S) \leqslant 1 \quad \forall\, a \in L \quad \cdots \quad (p^a) \tag{2.25}$$

$$x_i(S) \geqslant 0 \quad \forall\, S \in \Omega \quad \forall\, i \in B \tag{2.26}$$

Consider the dual of FO-NALP, which I call D-FO-NALP, presented as below:

$$\textbf{D-FO-NALP:} \qquad \min \sum_{i \in B} q_i + \sum_{a \in L} p^a \tag{2.27}$$

$$\text{subject to} \qquad q_i \geqslant v_i(S) - \sum_{a \in S} p^a \quad \forall\, i \in B,\ S \in \Omega \tag{2.28}$$

$$q_i \geqslant 0 \quad \forall\, i \in B \tag{2.29}$$

$$p^a \geqslant 0 \quad \forall\, a \in L \tag{2.30}$$

At optimality of D-FO-NALP, $q_i = \max\{v_i(S) - \sum_{a \in S} p^a, 0\}$ can be understood as the utility of bidder i if he is assigned set S and if he pays a price p^a for each item $a \in S$ he receives. The term $\sum_{a \in L} p^a$ is the auctioneer's revenue. However, unlike formulation LP in §2.2, the linear program FO-NALP does not possess the integrality property in general. Therefore, FO-NALP does not solve NAIP in general. Therefore, prices of the form $p^a, a \in L$, i.e., linear prices at the item level cannot bring about an allocation of indivisible items in a manner where the total welfare of all bidders is maximized when bidder valuations of a set S are non-additive. As I shall further discuss, pricing will need to be more complex (i.e., prices may have to depend on the identity of the bidders and would have to be set at the level of item sets instead of individual items) in order to bring about a social-welfare-maximizing and feasible allocation of the indivisible items.

2.3.2 Second-order Linear Programming Formulation

The second-order linear programming formulation corresponding to NAIP is constructed by modifying the constraint space of problem FO-NALP. Let Ψ be the set of all partitions of item set L. I call the following linear program SO-NALP.

$$\textbf{SO-NALP:} \qquad \max \sum_{S \in \Omega} \sum_{i \in S} v_i(S) x_i(S) \tag{2.31}$$

$$\text{subject to} \qquad \sum_{S \in \Omega} x_i(S) \leqslant 1 \quad \forall\, i \in B \quad \cdots \quad (q_i) \tag{2.32}$$

$$\sum_{i \in B} x_i(S) \leqslant \sum_{\psi \in \Psi : S \in \psi} y(\psi) \quad \forall\, S \in \Omega \quad \cdots \quad (\lambda_S) \tag{2.33}$$

$$\sum_{\psi \in \Psi} y(\psi) \leqslant 1 \quad \cdots \quad (\pi) \tag{2.34}$$

$$x_i(S) \geqslant 0 \quad \forall\, S \in \Omega \quad \forall\, i \in B \tag{2.35}$$

The dual of SO-NALP, that I call D-SO-NALP, is as below:

$$\textbf{D-SO-NALP:} \qquad \min \sum_{i \in B} q_i + \pi \tag{2.36}$$

$$\text{subject to} \qquad q_i \geqslant v_i(S) - \lambda_S \quad \forall\, i \in B,\ S \in \Omega \tag{2.37}$$

$$\pi \geqslant \sum_{S \in \psi} \lambda_S \quad \forall\, \psi \in \Psi \tag{2.38}$$

$$q_i \geqslant 0 \quad \forall\, i \in B \tag{2.39}$$

$$\lambda_S \geqslant 0 \quad \forall\, S \in \Omega \tag{2.40}$$

$$\pi \geqslant 0 \tag{2.41}$$

At optimality of D-SO-NALP, $\lambda_S, S \in \Omega$ is the price of bundle S, and $q_i =$

$\max\{v_(S) - \lambda_S, 0\}$ is the surplus of bidder i for bundle S, and π is the auctioneer's surplus. However, the linear program SO-NALP does not possess the integrality property. As a result, the SO-NALP does not solve NAIP. Therefore, the dual prices $\lambda_S, S \in \Omega$ do not support a feasible allocation of indivisible items that results in social-welfare maximization. Thus, a feasible allocation of indivisible items that results in social-welfare maximization is not supported even when the prices are of the form $\lambda_S, S \in \Omega$, i.e., the prices are set at the level of item subsets. As we shall see, more complex pricing at the level of both item subsets and the identity of the bidders is needed.

2.3.3 Third-order Linear Programming Formulation

Let θ be a feasible allocation of item bundles to bidders. Let Θ be the set of all feasible allocations of item bundles to bidder. The third-order linear programming formulation is constructed as follows. I call this problem TO-NALP.

$$\textbf{TO-NALP:} \qquad \max \sum_{S\in\Omega}\sum_{i\in B} v_i(S)x_i(S) \tag{2.42}$$

$$\text{subject to} \qquad \sum_{S\in\Omega} x_i(S) \leqslant 1 \quad \forall\ i \in B \quad \cdots \quad (q_i) \tag{2.43}$$

$$x_i(S) \leqslant \sum_{\theta\in\Theta:(i,S)\in\theta} y(\theta) \quad \forall\ i \in B,\ S \in \Omega \quad \cdots \quad (\delta_i(S)) \tag{2.44}$$

$$\sum_{\theta\in\Theta} y(\theta) \leqslant 1 \quad \cdots \quad (\pi) \tag{2.45}$$

$$x_i(S) \geqslant 0 \quad \forall\ i \in B,\ S \in \Omega \tag{2.46}$$

$$y(\theta) \geqslant 0 \quad \forall\ \theta \in \Theta \tag{2.47}$$

The first, second, and third constraints ensure that an item is not allocated to more than one bidder. The following linear program is the dual of TO-NALP. I call this D-TO-NALP.

$$\textbf{D-TO-NALP:} \qquad \min \sum_{i \in B} q_i + \pi \tag{2.48}$$

$$\text{subject to} \qquad q_i \geqslant v_i(S) - \delta_i(S) \quad \forall\ S \in \Omega \tag{2.49}$$

$$\pi \geqslant \sum_{\theta \in \Theta : (i,S) \in \theta} \delta_i(S) \quad \forall\ \theta \tag{2.50}$$

$$q_i \geqslant 0 \quad \forall\ i \in B \tag{2.51}$$

$$\delta_i(S) \geqslant 0 \quad \forall\ i \in B,\ S \in \Omega \tag{2.52}$$

At optimality of D-TO-NALP, we have that $q_i = \max\{v_i(S) - \delta_i(S), 0\}$. Here, a price of $\delta_i(S)$ is charged to bidder i who is assigned set S. π is the auctioneer's revenue. Linear program TO-NALP has integral solutions at optimality. Therefore, TO-NALP solves problem NAIP. Therefore, complex prices that are dependent on the item bundles and the identities of the bidders result in a social-welfare-maximizing allocation of the non-divisible items. Thus, prices that achieve a social-welfare-maximizing allocation of items always exist. However, the structure of such prices can be complex. Complex pricing structures make the practical implementation of an auction difficult if the aim of this auction is to discover such prices (and end with a social-welfare-maximizing allocation of items to bidders). Since the complexity of equilibrium prices is dependent on the nature of the valuation functions of the bidders, it is of academic interest to identify classes of valuation functions for which social-welfare-maximizing allocations are supported by simple price structures.

2.4 Computationally Tractable Instances of the Winner Determination Problems

The winner determination problem of the form NAIP equivalent to the maximum weighted set packing problem. As a result, such problems are computationally hard to solve in general. However, under certain conditions on the valuation functions

$v_i(S)$ for all $S \in \Omega$ for all $i \in B$, it is possible to solve the winner determination problem in polynomial time. For example, if the valuation function $v_i(S)$ were such that problem NALP $-$ a relaxation of the binary constraints on NAIP $-$ had integral solutions at optimality, then the winner determination problem can be solved in polynomial time. Other conditions, such as restrictions on the sizes or the structures of the bids, can also result in computationally tractable computation. de Vries and Vohra (2003), Rothkopf (1998), and Candogan et. al. (2015) present several such conditions that lead to computationally tractable solutions to the winner determination problems.

2.5 Primal-Dual Algorithms for the Winner Determination Problems

Primal-dual algorithms are a broad class of algorithms for combinatorial optimization problems. The problems are formulated in their primal and dual forms, and a primal-dual algorithm searches for primal and dual feasible solutions that satisfy the complementary slackness conditions. The broad structure of a primal-dual algorithm is as follows. A primal-dual algorithm

1. identifies a feasible dual solution and,
2. computes a feasible primal solution that minimizes violations of the complementary slackness conditions for the given feasible dual solution.
3. If the complementary slackness conditions are satisfied, the algorithm terminates.
4. If the complementary slackness conditions are not satisfied, the algorithm identifies a new feasible dual solution towards an optimal solution using information from the current primal solution and the complementary slackness conditions, and continues from step 2 with the updated dual solution.

Primal dual algorithms can be interpreted as a market mechanism. Step 1 can be interpreted as the auctioneer initializing a set of item prices. Step 2 can be interpreted as the bidders expressing their interest in the items for these prices. Step 3 can be interpreted checking if the market has cleared for these prices, i.e., if the bidders' interest in the items is such that a feasible allocation of items is possible at these prices. Step 4 can be interpreted as the auctioneer changing item prices in response to the interest expressed by the bidders to the old prices in a manner that leads to market clearance.

Notes

[1]Simple prices refer to prices set at the level of each item, and anonymous refer to the fact that item prices do not depend on the identities of the buyers.

Chapter 3

Optimal Bundling for Truthful Auctions

3.1 Introduction

Firms use online advertisements to promote their brands, products, and services on the Internet. They place their advertisements on various online communication channels such as social media websites (e.g., Facebook, Twitter, Instagram), electronic mails (Gmail), search engine webpages (e.g., Google's sponsored search), digital display advertising (e.g., advertisements on online news articles), and mobile advertising (e.g., in-app advertisements). Firms (henceforth referred to as advertisers) make payments to online platforms (such as Google and Facebook) to earn the rights to display their advertisements on the platforms. Auctions are commonly-used sale mechanisms for the sale of advertising positions on online media. The Vickrey-Clarke-Groves (VCG) (used by Facebook and Yandex[2]), and the Generalized Second-Price (GSP) (used by Google) are well-known auction formats for the sale of online advertising space (Varian et. al. 2014, Edelman et. al. 2007). We briefly review the working of the two mechanisms now.

Vickrey-Clarke-Groves (VCG) mechanism. The VCG mechanisms are a general class of truthful mechanisms. In such mechanisms, it is a dominant strategy for the bidders to report their valuation functions truthfully to the auctioneer. The VCG mechanism results in an allocation where the welfare of the bidders is maximized. This follows from the fact that each bidder is required to pay the externality he inflicts on the other bidders by participating in the auction.

Generalized Second Price (GSP) mechanism. The GSP is a mechanism often used for the sale of multiple items, particularly in the context of online adtervisement slots where there is clear ordering of the slots in terms of desirability. Unlike the VCG, the GSP is a non-truthful mechanism. It starts with each bidder presenting a single bid to the auctioneer. The bidder who bids the highest, receives the highest slot, the second-highest bidder receives the second-highest slot, and so on. I refer readers to Edelman et. al. (2007) for a detailed explanation of the working of the GSP.

In this chapter, I examine the problem of optimally partitioning the set of advertisement slots[3] into bundles before selling them via a VCG auction[4] to maximize the online platform's revenue from this VCG sale. In this study, I consider that (i) the advertisers' valuations of a bundle (multiple slots) is additive in the valuations of the constituent slots, (ii) an advertiser's valuation of any set of slots is positive, (iii) the advertisers are not budget constrained, and (iv) the online platform (auctioneer) is aware that $(i), (ii)$, and (iii) are true. It is noteworthy that there are several advantages to using a VCG format over other formats in settings involving the sale of multiple items. I refer readers to Varian et. al. (2014) for a detailed insight into the advantages of the VCG format for advertisement slot auctions on online platforms. When a group of slots is bundled, all the slots in a bundle are assigned to the same advertiser.

Consider the following example understand how bundling may improve VCG revenue for the online platform (under the aforementioned setting). Suppose there are a set of advertisement slots $\{a, b, c\}$ on a web page and three advertisers $\{1, 2, 3\}$ who wish to bid for these slots. Assume the online platform knows the advertisers' valuations of individual slots, as listed in Table 3.1a. For example, advertiser 2 values slot c at 84 units and advertiser 3 values slot b at 12 units. Under a VCG

	a	b	c
1	17	63	12
2	28	8	84
3	19	12	98

(a) Advertiser valuations for slots.

Partition (Bundling)	VCG Revenue
$\{a\}, \{b\}, \{c\}$	115
$\{a, b\}, \{c\}$	120
$\{b, c\}, \{a\}$	111
$\{a, c\}, \{b\}$	124
$\{a, b, c\}$	120

(b) VCG revenues from partitioning.

Table 3.1: Example of VCG revenue with bundling.

auction, the allocation of items to the advertisers for a given bundling of slots is such that it maximizes social welfare. Then, the advertisers make the VCG payments corresponding to the bundle they receive. The total payment received by the online platform is the VCG revenue. As shown in Table 3.1b, when the online platform allows separate sales, slot a is given to advertiser 2, slot b is given to advertiser 1, and slot c is given to advertiser 3. The social welfare, for this base case, is 28+63+98 = 189 units. Advertiser 2 pays 19 units, advertiser 1 pays 12 units, and advertiser 3 pays 84 units. Hence the online platform's VCG revenue is 19+12+84 = 115 units. However, bundling the items as $\{a, c\}, \{b\}$ provides the online platform with the highest revenue of 124 units. Under this bundling, slots a and c (i.e., the bundle of slots $\{a, c\}$) are given to advertiser 3, and slot b is given to advertiser 1. The social welfare from this allocation is 117+63 = 180 units. Advertiser 3 pays 112 units and advertiser 1 pays 12 units. Advertiser 2 pays nothing, since he is not assigned any slot.

Unlike the illustration discussed earlier, a practical setting poses two major challenges for the auctioneer[5], (i) how should the auctioneer elicit a bidder's true valuation of an item?, and (ii) how to optimally partition the set of items to maximize revenue under a VCG auction format? Without prior knowledge of the bidders' valuations, the chance that a chosen bundling would yield the highest revenue is almost negligible.

This leads us to the following research question: Is there a mechanism whereby

(i) bidders reveal their valuations truthfully to the auctioneer, and (ii) the auctioneer uses the valuations to optimally partition the set of items into bundles to maximize her revenue when she sells the bundles using a VCG auction format? These are important considerations, since, essentially, such a mechanism would yield the same revenue even if the auctioneer declared the bundles a priori, solicited bids for the bundles from the bidders, and then sold the bundles using a VCG auction. The goal of this chapter is to determine such a mechanism. As with most combinatorial problems with a similar structure, the number of possible partitions of a set of items, i.e., the *Bell number*, is extremely large. In particular, if there are n items, the number of ways of partitioning the set of n items into mutually exclusive collectively exhaustive sets is equal to $\mathcal{B}_n = \sum_{k=0}^{k=n-1} \binom{n-1}{k} \mathcal{B}_k$. As n increases, $\mathcal{B}_n$ can become very large leading to the complexity of computing the optimal bundling.

3.1.1 Summary of Our Main Contributions

My main contribution is that I identify, and analyze, a mechanism – the Pairwise Bundler Auction (PBA) – that allows an auctioneer to achieve truthful revelation and optimally partition the set of items before sale such that her VCG revenue is maximized. I reduce the optimal bundling problem – a bilevel, cubic, binary optimization problem – to a mixed binary integer programming formulation, and develop a Benders decomposition based algorithm to solve it. Additionally, as part of our discussion of the PBA mechanism, I present a class of binary integer, quadratic programming formulation, whose continuous relaxation guarantees (binary) integer solutions. This result is of general interest for a class of quadratic semi-assignment problems. Finally, using numerical experiments, I show that the revenue benefit of bundling is considerably higher when the ratio of the number of items to the number of bidders is high.

The rest of the chapter is organized as follows. In §3.2, I review relevant literature. I also explain our contribution to literature in detail. In §3.3, I describe the PBA auction format and present the optimal bundling problem, a bilevel binary cubic program, to identify the revenue-maximizing bundling, given the bidders' bid reports. In §3.4, I show that the PBA is truthful and present some technical results. In §3.5, I simplify the optimal bundling problem to a single-level binary linear integer program. In §3.6, I describe how Benders decomposition can be used as a solution technique to compute the optimal bundling. In §3.7, we present numerical results, and discuss insights from the numerical runs. In §3.8, I summarize the chapter and discuss future research.

3.2 Literature Review

Finding the optimal mechanism for the sale of multiple items to multiple buyers is seen as a difficult problem in auction theory literature (Tang et. al. 2012, Sandholm et. al. 2004). Myerson (1981) solved the problem for the single-item setting. Literature has several pieces of work on improving revenue using various mechanisms from practice such as posted prices, reserve-prices, and bundling (Chawla et. al. 2010, Balcan et. al. 2008, Guruswami et. al. 2005, Tang et. al. 2012). In this chapter, I examine the use of bundling as a means to improve revenue in a multi-item VCG sale setting. Bundling as a revenue-enhancement tool for the sale of multiple items has been studied in the contexts of bundle pricing and bundling in auctions. Here, I review literature in these contexts.

3.2.1 Literature on Bundle Pricing

Bundling has been studied in literature in the context of bundle pricing, i.e., the seller would decide on the bundles she would sell their prices. Some the earliest pieces of work in the area of bundling pricing is that of Adams et. al. (1976). Other later pieces of literature include Guiltinan (1987) and Hanson et. al. (1990). Taking note of the fact that computing optimal bundle compositions is intractable in general, bundling literature has discussed other forms of bundle-pricing schemes. For example, Hitt et. al. (2005), Wu et. al. (2008), and Wu et. al. (2019)

discuss a bundling framework where the buyers are allowed to choose a bundling of their choice based on the prices set by the seller. Chu et. al. (2011) discuss a bundling framework where the price of a bundle depends on the size of the bundle. My setting is different from these papers in the sense that I study an auction setting, i.e., bundle prices are determined by an auction.

3.2.2 Literature on Bundling in Auctions

One of the earliest pieces of literature in bundling in an auction setting is due to Palfrey (1983). One of his key results show that when there are two bidders in a second-price auction setting, bundling the items together would maximize the auctioneer's revenue. Sandholm et. al. (2004) study a class of combinatorial auctions called the virtual valuation combinatorial auctions (VVCA) in which the VCG is used on an affine transformation of the bidders' valuations. In this chapter, the authors propose computational solutions to enhance revenue in the combinatorial auction setting using the VVCA model. Subramaniam et. al. (2009) study bundling in a two-item second-price auction setting where the items may be complements or substitutes. Tang et. al. (2012) discuss bundling in a two-item setting where the valuation of the bundle of items is additive, and derive expressions for the optimal reserve price in the mixed bundling auction of the two items. The extant literature has also studied formats wherein the auctioneer grants exclusivity in the positioning of advertisements on webpages (Sayedi et. al. 2018). Bidders desiring exclusivity in advertising may express their interest in being the only advertiser on an entire webpage or on a part of the webpage (Constantin et. al. 2011, Sayedi et. al. 2018, Bhargava et. al. 2019). Bundling a set of slots and granting exclusivity on a set of slots are similar to the extent that the set slots are to be assigned to one bidder only.

Another related paper is that of Jehiel et. al. (2007). Here, the authors introduce a class of dominant-strategy auctions that assigns a weight c to the various partitions of the set of items on sale. These weights result in some partitions being chosen with higher probability than the others. The analysis in the paper revolves around finding an optimal value of c that maximizes expected revenue. A downside to using expected revenue as a criterion is that it may yield arbitrarily bad results when the bidders' valuations are realized since the bundling parameter c is decided *before* the bids are called for based on the prior distributions of the valuations. In our view, this is a limitation.

I present several results apart from extending the results in Jehiel et. al. (2007). First, I propose a class of truthful auction formats called the Pairwise Bundler Auctions (PBA). Using the PBA, the auctioneer can compute the optimal bundling *after* the bidders' valuations are realized and reported to the auctioneer. Therefore, the auctioneer does not need the knowledge of the distributions of the bidders' valuations of the items to compute the optimal bundling. Importantly, under this mechanism, the bidders are incentivized to truthfully report their valuations even though the auctioneer uses their bids to partition the set items into bundles *before* eventually selling them.

Second, I present a binary integer programming formulation to compute the optimal bundling of items. I obtain this formulation by reducing a bilevel, cubic, binary optimization problem to a linear binary integer programming problem. This model structure allows us to use a Benders decomposition solution approach, where the master problem is a binary integer program and the sub-problem is a linear program. Third, I describe how the dual variables corresponding to the linear program in the sub-problem in the final iteration can be interpreted as the marginal revenue benefit of having a given pair of items in the same bundle. Fourth, I present results from numerical experiments to show how the usefulness of bundling as a lever to improve revenue is the highest when there are many more items than there are bidders. I also discuss the impact of higher competition on the usefulness of bundling as a revenue-improvement lever. Another insight from our approach is that bundling the items is, in some sense, akin to manipulating the negative externalities that one bidder imposes on the others (Belloni et. al. 2017). Manipulating the externalities to make them take high values is identical to giving the bidders allocation exclusivity. Thus, besides contributing to literature on bundling in auctions, I add to the broader literature around revenue-enhancement mechanisms for the sale of multiple items to multiple buyers. In addition, I also present a class of linearly-constrained binary quadratic optimization problems whose relaxations provide integral solutions at optimality. This finding may be useful for future research into assignment problems involving negative externalities.

3.3 Pairwise Bundler Auctions (PBAs)

I begin this section with definitions of a few parameters and decision variables. I then describe the sequence of events, followed by the optimization model for the PBA.

3.3.1 Preliminaries

Let L represent a set of heterogeneous (indivisible) items to be auctioned, and B represent the set of bidders interested in acquiring some or all items in L. While a bidder may receive more than one item, each item is sold to at most one bidder. Each bidder $i \in B$ has a private value of u_i^a, $u_i^a > 0$, for item $a \in L$.

Assumption 3.1 *For each a, $a \in L$, bidder i's valuation u_i^a is sampled from a continuous probability distribution with finite support.*

As a result of Assumption 3.1, no bidder can exactly guess another bidder's valuation with non-zero probability. From a practical standpoint, it is almost impossible for a bidder, or the auctioneer, to guess a bidder's valuation exactly. If it were possible to do so, the auctioneer could simply post a price equal to the highest valuation for an item or a bundle and capture all of the bidder's surplus.

The decision variable x_i^a equals 1, if bidder i receives item a, and is zero otherwise. Bidder i's valuation for an allocation $\mathbf{x} = < x_i^a, i \in B, a \in L >$, $v_i(\mathbf{x})$, can be written as

$$v_i(\mathbf{x}) = \sum_{a \in L} u_i^a x_i^a, \tag{3.1}$$

where $BL = B \times L$ (the cartesian product of the sets B and L). The total social welfare of the bidders is given by

$$\sum_{i \in B} v_i(\mathbf{x}) = \sum_{(i,a) \in BL} u_i^a x_i^a. \tag{3.2}$$

The allowable allocations $\mathbf{x}$ are those allocations where $\sum_{i \in B} x_i^a = 1$ is satisfied for all $a \in L$ (it is an equality constraint since $u_i^a > 0$ implies that all items get assigned to a bidder).

3.3.2 The PBA

The sequence of events in the PBA is as shown in Figure 3.1. The sequence contains

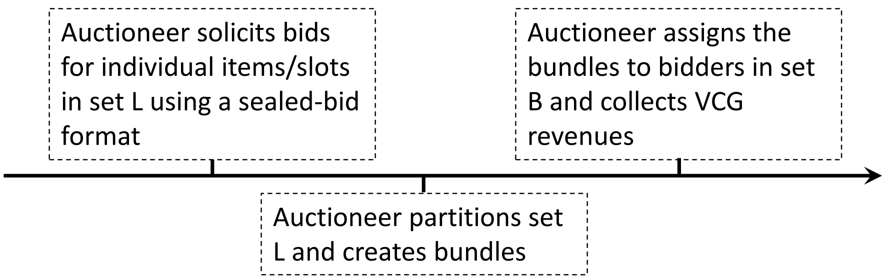


Figure 3.1: Sequence of events.

three important stages/steps.

Step 1: Each bidder $i \in B$ bids an amount $\hat{u}_i^a$ for each item $a \in L$ using a sealed-bid format.

Step 2: The auctioneer creates bundles by partitioning the set L using the bids $\hat{u}_i^a, \quad (i,a) \in BL$.

Step 3: She sells these bundles to individual bidders using the VCG auction format.

Note that $\hat{u}_i^a$ may or may not be equal to u_i^a for all $(i,a) \in BL$. The auctioneer's problem in Step 2 is to partition L in such a way that she maximizes her revenue from the VCG auction of the item bundles in Step 3. Thus, the auctioneer must make two levels of decisions simultaneously, (i) identify an optimal bundling of the item set, and (ii) identify the corresponding allocation of items to individual bidders. The PBA model formulation, described in §3.3.3, achieves both.

3.3.3 The PBA Optimization Model

I describe the components involved in building the auctioneer's optimization model in Step 2 that simultaneously partitions L optimally and determines optimal allocations of items to bidders in set B, after receiving sealed bids $\hat{u}_i^a, (i, a) \in BL$. Let the decision vector

$$\lambda =< \lambda^{ab},\ a \neq b,\ a, b \in L >, \tag{3.3}$$

be defined such that (a) $\lambda^{aa} = 0$, (b) $\lambda^{ab} \in \{0, 1\}$, and (c) $\lambda^{ab} = \lambda^{ba}\ \forall\ a\ , b \in L,\ a \neq b$. Essentially, λ represents a partition of the item set L in the following manner. For all $a, b \in L, b \neq a$, $\lambda^{ab} = 1$ if items a and b belong to the same bundle, and $\lambda^{ab} = 0$ if items a and b do not belong to the same bundle. The components of λ, of the form λ^{ab} where $a, b \in L, b \neq a$, are decision variables that model whether or not two items a and b belong to the same bundle.

Winner determination under partitioning (bundling) λ. Under any given partition, λ, the auctioneer conducts a VCG auction. The winners the VCG auction are obtained by solving the binary-integer optimization model (3.4) – (3.6), with decision vector $\mathbf{x}$. This optimization problem identifies the allocation that maximizes total value of the bids under bundling λ.

$$\max_{\mathbf{x}} \quad H(\mathbf{x}; \lambda) = \sum_{(i,a) \in BL} \Big[\hat{u}_i^a - \sum_{\substack{b \in L \\ b \neq a}} M^{ab} \lambda^{ab}\Big] x_i^a + \sum_{i \in B} \sum_{a \in L} \sum_{\substack{b \in L \\ b \neq a}} M^{ab} \lambda^{ab} x_i^a x_i^b, \tag{3.4}$$

$$\text{s.t.} \qquad \sum_{i \in B} x_i^a = 1 \quad \forall a \in L, \tag{3.5}$$

$$x_i^a \in \{0, 1\} \quad \forall \quad (i, a) \in BL. \tag{3.6}$$

In (3.4), the parameters M^{ab} for all $a, b \in L$ and for all $i \in B$ are "penalty" parameters such that

$$M^{ab} = \begin{cases} M & \text{if } a \neq b, \\ \\ 0 & \text{if } a = b, \end{cases} \tag{3.7}$$

where M is a suitably large positive value.

In any feasible solution, $x_i^a = 1$ implies that bidder i is assigned item a. If $x_i^a = 0$, it implies that bidder i is not assigned item a. The optimal allocation $\mathbf{x}^*$ is such that each bundle in the partition represented by λ is assigned to exactly one bidder. Notice that if $\lambda^{ab} = 1$ and $\lambda^{cd} = 0$ for all $c \neq a \neq b, d \neq a \neq b$ for some items a and b in L, then the allocation vector $\mathbf{x}^*$ that maximizes $H(\mathbf{x}; \lambda)$ is such that items a and b are assigned to the same bidder. This is because the objective function (3.4) incurs a large penalty from making an allocation where $x_i^a = 1$ and $x_j^b = 1$ where $i \neq j$. However, when $x_i^a = x_i^b = 1$ for some i, the terms $-M^{ab}\lambda^{ab}x_i^a$ and $+M^{ab}\lambda^{ab}x_i^a x_i^b$ cancel each other in the objective function (3.4), i.e. no penalty is incurred. Likewise, the terms $-M^{ba}\lambda^{ba}x_i^b$ and $+M^{ba}\lambda^{ba}x_i^b x_i^a$ cancel each other in the objective function (3.4). In effect, setting $\lambda^{ab} = 1$ yields an allocation of the form $x_i^a = x_i^b = 1$ for some $i \in B$. Essentially, when items a and b are bundled ($\lambda^{ab} = 1$), they cannot be allocated separately. Likewise, setting $\lambda^{ab} = 0$ allows for the sale of items a and b to two different bidders, say i and j, as there is no penalty from making such an assignment. This is because $M^{ab}\lambda^{ab}x_i^a = 0$ and $M^{ab}\lambda^{ab}x_i^a x_i^b = 0$ since $\lambda^{ab} = 0$. Thus, the item bundles (in the partition represented by λ) are assigned to the bidders in such a way that the total bid value is maximized and all items bundled together are assigned to exactly one bidder. Next, I describe the VCG revenue computation, given a partition λ.

Computing the VCG revenue under bundling λ. The first step to computing VCG revenue under bundling λ is to solve the total bid value maximization problem without bidder i, represented by the following optimization problem with the binary decision variables $\mathbf{y}_{-i} =< y_l^{a,-i}, l \in B, l \neq i, a \in L >$, for all bidders $i \in B$.

$$\max H_{-i}(\mathbf{y}^*_{-i}; \lambda) = \max_{\mathbf{y}_{-i}} \sum_{\substack{(l,a)\in BL \\ l\neq i}} \Big[\hat{u}_l^a - \sum_{\substack{b\in L \\ b\neq a}} M^{ab}\lambda^{ab}\Big] y_l^{a,-i} + \sum_{\substack{l\in B \\ l\neq i}} \sum_{a\in L} \sum_{\substack{b\in L \\ b\neq a}} M^{ab}\lambda^{ab} y_l^{a,-i} y_l^{b,-i}, \tag{3.8}$$

$$\text{s.t.} \qquad \sum_{\substack{l\in B \\ l\neq i}} y_l^{a,-i} = 1 \quad \forall \;\; a \in L, \tag{3.9}$$

$$y_l^{a,-i} \in \{0,1\} \quad \forall \;\; (l,a) \in BL, l \neq i. \tag{3.10}$$

Here, $y_l^{a,-i} = 0$ if bidder l is assigned item a and $y_l^{a,-i} = 0$ if bidder l is not assigned item a. This optimization problem is structurally identical to the model described in (3.4) – (3.6). The second step is to compute the function $H_{-i}(\mathbf{x}^*; \lambda)$ for each bidder $i \in B$ as follows.

$$H_{-i}(\mathbf{x}^*; \lambda) = \sum_{(l,a)\in BL} \Big[\hat{u}_l^{a,-i} - \sum_{\substack{b\in L \\ b\neq a}} M_l^{ab,-i}\lambda^{ab}\Big] x_l^{a*} + \sum_{l\in B} \sum_{a\in L} \sum_{\substack{b\in L \\ b\neq a}} M_l^{ab,-i}\lambda^{ab} x_l^{a*} x_l^{b*} \tag{3.11}$$

where $\mathbf{x}^*$ is the optimal solution to (3.4)) – (3.6) and for all $a \in L$ and $l, i \in B$,

$$\hat{u}_l^{a,-i} = \begin{cases} \hat{u}_l^a & \text{if } l \neq i, \\ 0 & \text{if } l = i, \end{cases} \tag{3.12}$$

and,

$$M_l^{ab,-i} = \begin{cases} M^{ab} & \text{if } l \neq i, \\ 0 & \text{if } l = i. \end{cases} \tag{3.13}$$

The quantity $H_{-i}(\mathbf{x}^*; \lambda)$ is the total bid value of all bidders excluding i under the optimal allocation $\mathbf{x}^*$ for bundling λ. The quantity $H_{-i}(\mathbf{x}^*; \lambda) - H_{-i}(\mathbf{y}^*_{-i}; \lambda)$ represents the magnitude of the "externality" imposed by bidder i on all other bidders by virtue of his participation in the auction with bundling λ. Therefore, under the VCG auction, bidder i is charged an amount equal to the externality he imposes, i.e., he is charged an amount $H_{-i}(\mathbf{y}^*_{-i}; \lambda) - H_{-i}(\mathbf{x}^*; \lambda)$. The auctioneer's revenue from the VCG auction under partition λ can be computed as

$$R(\lambda) = \sum_{i \in B} \Big[H_{-i}(\mathbf{y}^*_{-i}; \lambda) - H_{-i}(\mathbf{x}^*; \lambda) \Big]. \tag{3.14}$$

Thus, for a given bundling λ, the bidder who is assigned a bundle is the bidder who values it the most, and this bidder pays the second-highest bid for the bundle. This is because the bundles in the partition represented by λ are sold in simultaneous VCG auctions (since bundle valuations are additive), and for each bundle, the VCG payment corresponds to the second-highest bid for the bundle (Krishna 2002, Ausubel 2006). Using the illustration in §5.1, in Table 3.2, I show how the various bundlings, encoded using the parameters $\lambda^{ab}, a, b \in L, a \neq b$ when $L = \{a, b, c\}$, induce the appropriate VCG revenues.

Bundling	λ^{ab}	λ^{bc}	λ^{ac}	Revenue
$\{a\}, \{b\}, \{c\}$	0	0	0	115
$\{a, b\}, \{c\}$	1	0	0	120
$\{b, c\}, \{a\}$	0	1	0	111
$\{a, c\}, \{b\}$	0	0	1	124
$\{a, b, c\}$	1	1	1	120

Table 3.2: Partitions and corresponding VCG revenues.

Next, putting together all the aforementioned optimization models, I describe the auctioneer's optimization problem in Step 2 that simultaneously determines the optimal bundling and corresponding winner allocations.

The optimal bundling problem (OBP). The auctioneer's optimization problem in Step 2 of the PBA is to identify $\lambda^* = \arg\max_\lambda R(\lambda)$ while simultaneously determining the winners/allocations. I present problem OBP below. OBP is the expanded representation of the optimization problem $\max_\lambda R(\lambda)$ described in (3.14). As mentioned earlier, the decision variables $\lambda =< \lambda^{ab}, a, b \in L >$ correspond to the Step 2 decisions of identifying a bundling, while the variables $\mathbf{x}$ and $\mathbf{y} =< y_l^{a,-i}, l \in B, l \neq i, a \in L, i \in B >$ correspond to the Step 3 decisions of allocations and VCG payments under the bundling identified in Step 2.

$$\textbf{OBP:} \qquad \max_{\lambda} \sum_{i \in B} \left[\sum_{\substack{l \in B \\ l \neq i}} \sum_{a \in L} \left[\hat{u}_l^a - \sum_{\substack{b \in L \\ b \neq a}} M^{ab} \lambda^{ab} \right] y_l^{a,-i} + \sum_{\substack{l \in B \\ l \neq i}} \sum_{a \in L} \sum_{\substack{b \in L \\ b \neq a}} M^{ab} \lambda^{ab} y_l^{a,-i} y_l^{b,-i} \right]$$

$$- \sum_{i \in B} \left[\sum_{l \in B} \sum_{a \in L} \left[\hat{u}_l^{a,-i} - \sum_{\substack{b \in L \\ b \neq a}} M_l^{ab,-i} \lambda^{ab} \right] x_l^{a*} + \sum_{l \in B} \sum_{a \in L} \sum_{\substack{b \in L \\ b \neq a}} M_l^{ab,-i} \lambda^{ab} x_l^{a*} x_l^{b*} \right], \tag{3.15}$$

$$\text{s.t.} \qquad \sum_{\substack{l \in B \\ l \neq i}} y_l^{a,-i} = 1 \quad \forall \;\; a \in L \quad \forall i \in B, \tag{3.16}$$

$$\mathbf{x}^* = \arg\max_{\mathbf{x}} H(\mathbf{x}; \lambda) \text{ subject to } \sum_{i \in B} x_i^a = 1 \quad \forall \;\; a \in L, x_i^a \in \{0, 1\} \quad \forall \;\; (i, a) \in BL, \tag{3.17}$$

$$y_l^{a,-i} \in \{0, 1\} \quad \forall \;\; (l, a) \in BL, l \neq i, \quad \forall i \in B, \tag{3.18}$$

$$\lambda^{ab} = \lambda^{ba} \quad \forall \;\; a \in L, b \in L, b \neq a, \tag{3.19}$$

$$\lambda^{aa} = 0 \quad \forall \;\; a \in L, \tag{3.20}$$

$$\lambda^{ab} \in \{0, 1\} \quad \forall \;\; a \in L, b \in L, b \neq a. \tag{3.21}$$

The objective function (3.15) is the expansion of $R(\lambda)$ defined in equation (3.14). Constraints (3.16) and (3.18) follow from constraints (3.9) and (3.10). Constraint (3.17) follows from the winner determination model in (3.4) – (3.6). Constraints (3.19), (3.20), (3.21) follow from the definition of λ. Thus, identifying λ^* that maximizes OBP is equivalent to identifying the bundling that maximizes the auctioneer's revenue from the VCG auction of the item bundles in Step 3 of the PBA.

I point out that there is no need to include "consistency" constraints. For example, if items a, b, c belong to the same bundle, there is no need to specify constraints of the form "if $\lambda^{ab} = \lambda^{bc} = 1$, then $\lambda^{ac} = 1$". This is because setting $\lambda^{ab} = \lambda^{bc} = 1$ is sufficient to ensure that items a, b, c are assigned to the same bidder. Setting $\lambda^{ab} = 1$ ensures that the objective function is penalized if items a and b are assigned to different bidders. Likewise, setting $\lambda^{bc} = 1$ ensures that the objective function is penalized if items b and c are assigned to different bidders. Consequently, items

a, b, c will end up being assigned to the same bidder. In conclusion, if a, b, c belong to the same bundle, then $\lambda^{ab} = \lambda^{bc} = 1, \lambda^{ac} = 0/1$ ensures the same objective function value for OBP.

3.4 PBA Model Analysis

In this section, I present important structural results related to the optimization model OBP developed in §3.3. I begin with the inner binary integer optimization problem (3.4) – (3.6), i.e., constraint (3.17), when λ is fixed. It is noteworthy that the model is a binary-integer, quadratic programming, problem with linear semi-assignment constraints – an NP-Hard problem in general.

Consider the following continuous version of the quadratic programming problem (3.4) – (3.6) wherein decision variables $x_i^a, (i, a) \in BL$ are relaxed to permit fractional values at optimality.

$$\max_{\mathbf{x}} H(\mathbf{x}; \lambda) = \sum_{(i,a)\in BL} \Big[\hat{u}_i^a - \sum_{\substack{b\in L \\ b\neq a}} M^{ab}\lambda^{ab}\Big]x_i^a + \sum_{i\in B}\sum_{a\in L}\sum_{\substack{b\in L \\ b\neq a}} M^{ab}\lambda^{ab}x_i^a x_i^b, \tag{3.22}$$

$$\text{s.t.} \qquad \sum_{i\in B} x_i^a = 1 \quad \forall \;\; a \in L, \tag{3.23}$$

$$0 \leqslant x_i^a \leqslant 1 \quad \forall (i, a) \in BL. \tag{3.24}$$

I show that the continuous relaxation continues to guarantee binary integer optimal solutions in Theorem 3.1.

Theorem 3.1 *The optimal solution, $\mathbf{x}^*$, to problem* (3.22) – (3.24) *is integral.*

Proof of Theorem 3.1.

I first prove a technical result, and show that the statement of Theorem 3.1 is true using this result. Let

$$Q(\mathbf{x}) = \sum_{(i,a)\in BL} u_i^a x_i^a - \sum_{(i,a)\in BL}\sum_{(j,b)\in BL} w_{ij}^{ab} x_i^a x_j^b.$$

Here, the parameter w_{ij}^{ab} is defined as follows:

$$w_{ij}^{ab} = \begin{cases} 0 \text{ if } a = b \text{ or } i = j, \\ \geqslant 0 \text{ otherwise.} \end{cases}$$

Now, consider the following optimization problem

$$\max_{\mathbf{x}} Q(\mathbf{x}) = \max_{\mathbf{x}} \sum_{(i,a)\in BL} u_i^a x_i^a - \sum_{(i,a)\in BL} \sum_{(j,b)\in BL} w_{ij}^{ab} x_i^a x_j^b, \tag{3.25}$$

$$\text{s.t.} \qquad \sum_{i\in B} x_i^a = 1, \quad \forall \;\; a \in L, \tag{3.26}$$

$$0 \leqslant x_i^a \leqslant 1 \quad \forall \;\; i \in B \quad \forall \;\; a \in L. \tag{3.27}$$

I claim that the solution $\mathbf{x}^*$ to (3.25) – (3.27) is integral. I now prove the claim:

Assume, to the contrary, that the optimal solution $\mathbf{x}^*$ to (3.25) – (3.27) is not integral. This means that for some (i, a), x_i^{a*} is fractional. Suppose x_p^{k*} is fractional. This implies that variable x_v^{k*} is also fractional for some $v \neq p$. This follows from constraint (3.26). In other words, we have that item k is fractionally allocated to bidders p and v.

The component x_p^{k*} of $\mathbf{x}^*$ contributes an amount $Q(x_p^{k*}) = \left(u_p^k - \sum_{(j,b)\in BL} w_{pj}^{kb} x_j^{b*}\right) x_p^{k*}$ to the objective function $Q(\mathbf{x}^*)$ and the component x_v^{k*} contributes an amount $Q(x_v^{k*}) = \left(u_v^k - \sum_{(j,b)\in BL} w_{vj}^{kb} x_j^{b*}\right) x_v^{k*}$ to the objective function $Q(\mathbf{x}^*)$. Let $Z(x_p^{k*}) = \left(u_p^k - \sum_{(j,b)\in BL} w_{pj}^{kb} x_j^{b*}\right)$. Similarly, define $Z(x_v^{k*}) = \left(u_v^k - \sum_{(j,b)\in BL} w_{vj}^{kb} x_j^{b*}\right)$. I point out that the coefficient of the term of the form $x_p^{k*} x_v^{k*}$ is zero. Therefore, such a term does not exist.

I now show that the optimality of x_p^{k*} and x_v^{k*} implies that $Z(x_p^{k*}) = Z(x_v^{k*})$.

Case 1. Suppose $Z(x_p^{k*}) \geqslant Z(x_v^{k*})$. Consider a solution $\hat{\mathbf{x}}$ where component $\hat{x}_p^k = x_p^{k*} + x_v^{k*}$, $\hat{x}_v^k = 0$ and $\hat{x}_i^a = x_i^{a*}$ for all $(i, a) \in BL, (i, a) \neq (p, k), (i, a) \neq (v, k)$. Now $Q(\hat{\mathbf{x}}) = Q(\mathbf{x}^*) + Z(x_p^{k*})x_v^{k*} - Z(x_v^{k*})x_v^{k*} \geqslant Q(\mathbf{x}^*)$ since $Z(x_p^{k*}) \geqslant Z(x_v^{k*})$.

Case 2. Suppose $Z(x_v^{k*}) \geqslant Z(x_p^{k*})$. Consider a solution $\hat{\mathbf{x}}$ where component $\hat{x}_v^k = x_v^{k*} + x_p^{k*}$, $\hat{x}_p^k = 0$ and $\hat{x}_i^a = x_i^{a*}$ for all $(i, a) \in BL, (i, a) \neq (p, k), (i, a) \neq (v, k)$.

Now $Q(\hat{\mathbf{x}}) = Q(\mathbf{x}^*) + Z(x_v^{k*})x_p^{k*} - Z(x_p^{k*})x_p^{k*} \geqslant Q(\mathbf{x}^*)$ since $Z(x_v^{k*}) \geqslant Z(x_p^{k*})$.

Therefore, $Z(x_p^{k*}) = Z(x_v^{k*})$.

Now, I define a solution $\hat{\mathbf{x}}$ where $\hat{x}_p^k = x_p^{k*} + x_v^{k*}, \hat{x}_v^k = 0, \hat{x}_i^a = x_i^{a*}$ for all (i,a). Note that the number of fractional components of $\hat{\mathbf{x}}$ is one less than the number of fractional components of $\mathbf{x}^*$, and can have two less fractional components if $\hat{x}_p^k = 1$. Since $Z(x_v^{k*}) = Z(x_p^{k*})$, we have that $Q(\hat{\mathbf{x}}) = Q(\mathbf{x}^*)$. If $\hat{x}_p^k < 1$, there must exist some fractional variable $\hat{x}_v^k$. I can then define another solution $\bar{\mathbf{x}}$ where $\bar{x}_p^k = \hat{x}_p^k + \hat{x}_v^k, \bar{x}_v^k = 0, \bar{x}_i^a = \hat{x}_i^a$ for all (i,a) in a manner similar to the definition of $\hat{\mathbf{x}}$ and repeat the same arguments with $\bar{\mathbf{x}}$ as I did with $\hat{\mathbf{x}}$. Say I arrive at a solution $\mathbf{y}$ where $y_p^k = 1, y_v^k = 0$ for all $p, p \neq v$ and $y_i^a = x_i^{a*}$ for all (i,a).

Now, starting with the solution $\mathbf{y}$, I repeat the above arguments for each item that is fractionally allocated to two or more bidders. With this, I arrive at a solution where the variables at optimality are either 0 or 1.

I now derive a special-case for our setting. Consider the term $\sum_{(i,a)\in BL} \sum_{(j,b)\in BL} w_{ij}^{ab} x_i^a x_j^b$ in (3.25). I set $w_i^{ab} = w_{ij}^{ab}$ for all $j \in B, j \neq i$.

$$
\begin{aligned}
\sum_{(i,a)\in BL} \sum_{(j,b)\in BL} w_{ij}^{ab} x_i^a x_j^b &= \textstyle\sum_{(i,a)\in BL} x_i^a \Big[\sum_{(j,b)\in BL} w_{ij}^{ab} x_j^b \Big], \\
&= \textstyle\sum_{(i,a)\in BL} x_i^a \Big[\sum_{b\in L} \sum_{j\in B} w_{ij}^{ab} x_j^b \Big], \\
&= \textstyle\sum_{(i,a)\in BL} x_i^a \Big[\sum_{b\in L} w_i^{ab} \sum_{\substack{j\in B \\ j\neq i}} x_j^b \Big], \\
&= \textstyle\sum_{(i,a)\in BL} x_i^a \Big[\sum_{\substack{b\in L \\ b\neq a}} w_i^{ab} (1 - x_i^b) \Big].
\end{aligned}
$$

Using constraint (3.26) I get,

$$
\begin{aligned}
\textstyle\sum_{\substack{j\in B \\ j\neq i}} x_j^b &= 1 - x_i^b, \\
&= \sum_{(i,a)\in BL} \sum_{\substack{b\in L \\ b\neq a}} w_i^{ab} x_i^a (1 - x_i^b).
\end{aligned}
$$

Thus, when $w_{ij}^{ab} = w_i^{ab}$ for all $j \in B, j \neq i$ for all $i \in B$, the objective function (3.25) is

$$\sum_{i\in B}\sum_{a\in L} u_i^a x_i^a - \sum_{i\in B}\sum_{a\in L}\sum_{\substack{b\in L\\ b\neq a}} w_i^{ab} x_i^a (1-x_i^b) \tag{3.28}$$

or,

$$\sum_{(i,a)\in BL}\Big[u_i^a - \sum_{\substack{b\in L\\ b\neq a}} w_i^{ab}\Big] x_i^a + \sum_{i\in B}\sum_{a\in L}\sum_{\substack{b\in L\\ b\neq a}} w_i^{ab} x_i^a x_i^b. \tag{3.29}$$

Consequently, by setting $w_{ij}^{ab} = w_i^{ab}$ for all $j \in B, j \neq i$ for all $i \in B$, I get the following quadratic optimization problem:

$$\max \sum_{(i,a)\in BL}\Big[u_i^a - \sum_{\substack{b\in L\\ b\neq a}} w_i^{ab}\Big] x_i^a + \sum_{i\in B}\sum_{a\in L}\sum_{\substack{b\in L\\ b\neq a}} w_i^{ab} x_i^a x_i^b \tag{3.30}$$

$$\text{s.t.} \quad \sum_{i\in B} x_i^a = 1 \quad \forall \;\; a \in L, \tag{3.31}$$

$$0 \leqslant x_i^a \leqslant 1 \quad \forall \;\; (i,a) \in BL. \tag{3.32}$$

Problem (3.22) – (3.24) is an instance of the above problem (3.30) – (3.32) where $w_i^{ab} = M^{ab}\lambda^{ab}$ for all $i \in B$ and $a, b \in L, b \neq a$. Thus, problem (3.22) – (3.24) has integral optimal solutions. Hence, Theorem 3.1 follows. ▫

It is noteworthy that *our proof of Theorem 3.1 is a more general result (applicable to quadratic programs modeling externalities and with linear semi-assignment constraints) which is of interest of its own.* Theorem 3.1 has implications for developing solution algorithms that guarantee optimal solutions. However, it is well known that the general continuous quadratic programs continue to remain NP-hard because the objective function is generally neither convex nor concave. But, in this case, upon closer inspection of the model in (3.22) – (3.24), it is readily noticeable that the model is structurally identical to the problem described in problem (1) in Page 756, §3.1 of Candogan et. al. (2015). Consequently, the continuous relaxation (3.22) – (3.24) can be solved using a linear program. I make use of the technique detailed in Appendix §3.9 to solve (3.22) – (3.24). As a consequence of this, instances of this model can be solved in polynomial time.

In addition to the integrality result in Theorem 3.1, any optimal solution to (3.22)

– (3.24), and consequently to the binary integer quadratic programming problem (3.4) – (3.6), satisfies the following necessary condition mentioned in Lemma 3.1.

Lemma 3.1 *Given a partition* λ*, define* $\eta(\mathbf{x}) = -\sum_{(i,a)\in BL}\sum_{\substack{b\in L\\ b\neq a}} M^{ab}\lambda^{ab}x_i^a(1-x_i^b)$. *If* $\mathbf{x}^*$ *is the optimal solution to* (3.22) *–* (3.24)*, then* $\eta(\mathbf{x}^*) = 0$ *in the expansion of* $H(\mathbf{x}^*;\lambda)$.

Proof of Lemma 3.1. If λ is such that $\lambda^{ab} = \lambda^{ba} = 1$ for some $a, b \in L, b \neq a$, then for some $i \in B$, we have that $x_i^{a*} = x_i^{b*} = 1$ at optimality. Thus, the terms $-M^{ab}\lambda^{ab}x_i^{a*}$ and $M^{ab}\lambda^{ab}x_i^{a*}x_i^{b*}$ cancel out since one is a negation of the other. If $\lambda^{ab} = 0$, then $-M^{ab}\lambda^{ab}x_i^a = M^{ab}\lambda^{ab}x_i^a x_i^b = 0$ for any value of $\mathbf{x}$. Thus, Lemma 3.1 holds true. □

Intuitively, Lemma 3.1 implies that the "big-M" terms do not figure in the optimal solution to (3.22) – (3.24), and consequently to the binary integer quadratic programming problem (3.4) – (3.6). The big-M penalty exists to influence the optimal winner allocation ($\mathbf{x}^*$) to the partition vector λ.

We now state the following Lemma that describe the auctioneer's bundling choices that the bidders would prefer the most.

Lemma 3.2 *Separate selling maximizes social welfare from the resulting allocation. A bidder would make a weakly higher surplus from a separate sale than from a sale after bundling in any form.*

Proof of Lemma 3.2. Separate selling maximizes social welfare since every bidder who values an item the most receives the item, whereas under bundling, a bidder may receive an item even if he does not value it the most, while the bidder valuing such an item the most does not receive it. As a result, a bidder would always prefer that the auctioneer choose separate selling. □

From Lemma 3.2, it appears that the bidders may have the incentive to misreport their bids in Step 1 to influence the auctioneer to conduct a separate sale of the items. However, as we shall show, truthful bidding in Step 1 is a weakly dominant strategy.

Proposition 3.1 *The sealed bids collected by the auctioneer in Step 1 of the PBA are truthful, i.e., truthful bidding is a weakly dominant strategy for all bidders in set B.*

Proof of Proposition 3.1. Step 1 of PBA is for the bidders to report their bids to the auctioneer. Step 2 is where the auctioneer would compute a value for λ, i.e., the auctioneer decides on an item bundling. Step 3 is where the auctioneer would allocate the bundles to the bidders and collect payments for the the bundle. It is important to note that in Step 2, the bids collected in the Step 1 are used to compute item bundlings, allocations, and payments that maximize the auctioneer's revenue from Step 3. We now present a Lemma.

Lemma 3.3 *Given a partition of items, which is unknown to the bidders, it is a dominant strategy for them to bid truthfully at the item level, so that their bid for any bundle is truthful, irrespective of the partition created by the auctioneer.*

Proof of Lemma 3.3. Let $\hat{u}_i^a, (i,a) \in BL$ be bidder i's reported bids for item $a \in L$ in Step 1. After the bidders report their bids to the auctioneer in Step 1, the auctioneer would consider one out of the $\mathcal{B}_{|L|}$ possible bundlings in Step 2 (here, $\mathcal{B}_{|L|}$ is the $|L|^{th}$ Bell number; Bell numbers were discussed in §5.1). Let P_A be one bundle of items in the bundling that the auctioneer decides in the second step of the PBA. According to the rules of the PBA, if bidder i wins bundle P_A, which happens if his bid for bundle P_A is the highest, his payment would be the second-highest bid for bundle P_A. These allocation and payments are computed in Step 3. Therefore, bidder i would find it weakly dominant to report his value for bundle P_A truthfully in Step 1., i.e., $\sum_{a \in P_A} \hat{u}_i^a = \sum_{a \in P_A} u_i^a$. The argument behind this assertion is as follows: Let $\hat{U}_i^A = \sum_{a \in P_A} \hat{u}_i^a$ for some i and let $U_i^A = \sum_{a \in P_A} u_i^a$. Let $\bar{U}^A =$ second-highest $_{j \in B} \sum_{a \in P_A} \hat{u}_j^a$.

1. If $\hat{U}_i^A > \bar{U}^A > U_i^A$, the bidder i wins bundle P_A, but pays $\bar{U}^A$. Since $U_i^A - \bar{U} \leqslant 0$, bidding $\hat{U}_i^A > U_i^A$ is not a rational decision for bidder i.

2. If $\bar{U}_i^A > \hat{U}_i^A$, then bidder i loses the auction for bundle P_A. He pays nothing, and gains nothing. Thus, his payoff is zero. Here $\hat{U}_i^A$ may be greater than, equal to, or lower than U_i^A.

3. If $\hat{U}_i^A > U_i^A > \bar{U}^A$, then bidder i wins the auction for bundle P_A and pays $\bar{U}^A$. Thus, any value of $\hat{U}_i^A > \bar{U}^A$ would have resulted in bidder i winning bundle P_A followed by paying $\bar{U}^A$.

Thus, it is a weakly dominant strategy for any bidder to bid his own valuation in Step 1 for any bundle P_A that may come about in Step 2. For any bundle $P_A \subseteq L$ and bidder i, the value of $\hat{u}_i^a$ should be such that $\hat{U}_i^A = U_i^A$.

From these arguments, it may appear that as long as $\hat{U}_i^A = U_i^A$, bidder i plays rationally irrespective of what the values of $\hat{u}_i^a$ are. However, we claim that it is important for the bidder that the bids be true at the item level, i.e., $\hat{u}_i^a = u_i^a$ for all $(i, a) \in BL$. Suppose bidder i reports $\hat{u}_i^a \neq u_i^a$ for all $a \in L$ in Step 1 of the PBA. Based on these reports, the auctioneer comes up with an item bundling in Step 2. Let $P_1, P_2, \cdots, P_k, \cdots, P_M$ be the item bundles in the bundling. Note that the bidders do not know what these item bundles are in Step 1. Let $U_i^k = \sum_{a \in P_k} u_i^a$ be bidder i's valuation for bundle P_k. Note, from the discussion above, that truthful bidding at the level of the bundle is weakly dominant.

In Step 3, if it turns out that $\sum_{a \in P_k} \hat{u}_i^a \neq U_i^k$, then bidder i's reported value for bundle P_k is not truthful. This will result in the following outcome in Step 3 of the PBA: (i) If bidder i values the bundle P_k the highest, he makes a non-negative surplus (i.e., positive surplus if he is assigned P_k, and zero surplus if he is not assigned P_k), and (ii) if bidder i does not value bundle P_k the highest, but $\sum_{a \in P_k} \hat{u}_i^a$ is the highest, he wins, and he makes a negative surplus on bundle P_k.

In Step 3, if it turns out that $\sum_{a \in P_k} \hat{u}_i^a = U_i^k$, then bidder i's reported for bundle P_k is truthful, and this results in the following outcome: (i) If bidder i values the bundle P_k the highest, he wins, and makes a non-negative surplus, and (ii) if bidder i does not value the bundle P_k the highest, he loses, and makes a zero surplus on bundle P_k.

Also, when all bidders $l \in B, l \neq i$ bid truthfully, i.e., $\hat{u}_l^a = u_l^a$ for all $l \in B, l \neq i$ and $a \in L$, the positive surplus that bidder i makes on any bundle $P_j, j \neq k$ under non-truthful bidding, if he wins bundle $P_j, j \neq i$, is equal to the positive surplus that bidder i makes on bundle $P_j, j \neq k$ under truthful bidding. Therefore, non-truthful reporting for P_k is weakly dominated by truthful reporting for P_k, since truthful reporting removes the possibility of making a negative surplus on winning P_k. Thus, the reported values $\hat{u}_i^a$ such that $\sum_{a \in P_k} \hat{u}_i^a = \sum_{a \in P_k} u_i^a$ for all k is when $\hat{u}_i^a = u_i^a$ for all $(i, a) \in BL$. This proves Lemma 3.3.

We now move onto the proof of Proposition 3.1 now by examining the bidders' incentives to bid truthfully. First, we conduct the analysis using two item bundles P_A and P_B. To start, assume that all bidders bid truthfully except bidder k. As part of the analysis that follows, we examine bidder k's incentives to be truthful and his incentives to misreport his bid.

Suppose bidder k bid truthfully. The following cases can arise.

1. Bidder k's value for P_{AB} is not the highest. Bidder k has the highest value for P_A. Bidder k does not have the highest value for P_B.
    (a) Suppose the bids are such that the auctioneer chooses to sell P_A and P_B separately. Then bidder k wins P_A, and does not win P_B. Bidder k makes a positive surplus from winning P_A, and makes zero surplus from P_B, since he does not win it.
    (b) Suppose the bids are such that the auctioneer chooses to sell bundle P_{AB}. Bidder k makes a surplus of zero since he does not win P_{AB}. Note that bidder k has not been able to win P_A despite having the highest value for P_A.
2. Bidder k's value for P_{AB} is not the highest. Bidder k's value for P_A is not the highest, and his value for P_B is not the highest.
    (a) Suppose the bids are such that the auctioneer chooses to sell P_A and P_B separately. Then bidder k makes a surplus of zero.

(b) Suppose the bids are such that the auctioneer chooses to sell P_{AB}. Then bidder k makes a surplus of zero.

3. Bidder k's value for P_{AB} is the highest. Bidder k's value for P_A is the highest. Bidder k's value for P_B is the highest.

   (a) Suppose the bids are such that the auctioneer chooses to sell P_A and P_B separately. The bidder k wins both P_A and P_B. He makes a positive surplus on both P_A and P_B.

   (b) Suppose the bids are such that the auctioneer chooses to sell P_{AB}. Bidder k wins P_{AB}, but makes a lower (but positive) surplus than he does in case 3(a).

4. Bidder k's value for P_{AB} is not the higehst, but his values for P_A and P_B are separately the highest.

   (a) This is an impossible case.

We now discuss bidder k's incentives to misreport his bids for each of these cases. Let the bidders' valuations for P_A and P_B be the following.

	P_A	P_B
1	a_1	b_1
2	a_2	b_2
$\vdots$	$\vdots$	$\vdots$
k	a_k	b_k
$\vdots$	$\vdots$	$\vdots$
$\|B\|$	$a_{\|B\|}$	$b_{\|B\|}$

For cases 2(a), 2(b), 3(a), and 3(b), if bidder k bids higher than his truthful bid on either P_A or P_B or both, his surplus is either going to be zero or is going to be non-negative. Therefore, for these cases, bidder k has no incentives to bid higher.

We now look at Cases 1(a) and 1(b) in detail. We begin with Case 1(a).

Case 1(a). From the assumptions of Case 1(a), we have that (i) $a_k+b_k < a_j+b_j$ for some j, (ii) $a_k > a_j$ for all j, (iii) $b_k < b_j$ for some j, and that (iv) second-highest report for P_A + second-highest report for P_B is greater than or equal to the second-highest report for P_{AB}, since the auctioneer prefers separate selling in Case 1(a). Under separate selling, bidder k makes zero surplus from P_B, and makes positive surplus from winning P_A.

By bidding higher on P_A or P_B, bidder k could make the auctioneer choose separate selling of P_A and P_B or bidder k could make the auctioneer choose to sell P_{AB}.

1. Under the misreported bids, suppose the auctioneer chooses to sell P_A and P_B separately, then bidder k would win P_A and P_B, but would make a negative surplus on P_B, while winning the same surplus on P_A as under truthful bidding. Thus, truthful bidding weakly dominates.

2. Under the misreported bids, suppose the auctioneer chooses to sell P_{AB}. Bidder k can then be the highest bidder on P_{AB}, but his surplus from winning P_{AB} will be lower than his surplus from truthful bidding. Thus, truthful bidding weakly dominates.

Thus, for Case 1(a), truthful bidding is weakly dominant. We now examine Case 1(b).

Case 1(b). From the assumptions of Case 1(b), we have that (i) $a_k+b_k < a_j+b_j$ for some j, (ii) $a_k > a_j$ for all j, (iii) $b_k < b_j$ for some j, and (iv) second-highest report for P_{AB} is greater than or equal to the second-highest report for P_A + second-highest report for P_B. Let $a_k = \alpha$. Let $b_k = \beta$. Let the second-highest report for P_A be γ. Let the second-highest report for P_B be δ. Let the highest bid for P_B be μ, and assume that this bid is from bidder s. Let bidder s's bid for P_A be ϵ.

From the case 1(b) assumption that bidder k does not win bundle P_{AB}, we have that

$$\alpha + \beta < \gamma + \delta < \epsilon + \mu \quad \text{for some bidder } s. \tag{3.33}$$

Also, we have that

$$\alpha > \gamma > \epsilon \text{ and } \mu > \delta > \beta \tag{3.34}$$

Suppose bidder k misreported his bid for P_B as τ instead of β, where $\tau > \mu$. It could result in the auctioneer choosing either selling P_A and P_B separately, or selling bundle P_{AB}. If the auctioneer chose to sell bundle P_{AB}, then misreporting is weakly dominated by truthful bidding. If the auctioneer chose to sell P_A and P_B separately, we have the following analysis: We have that

$$\alpha + \beta < \gamma + \delta < \epsilon + \mu \quad \text{for some bidder } s.$$

Subtracting ϵ from all sides, we have

$$\alpha + \beta - \epsilon < \gamma + \delta - \epsilon < \epsilon + \mu - \epsilon$$

Subtracting β from all sides, we have

$$\alpha + \beta - \epsilon - \beta < \gamma + \delta - \epsilon - \beta < \epsilon + \mu - \epsilon - \beta$$

$$\implies \alpha - \epsilon < \gamma + \delta - \epsilon - \beta < \mu - \beta$$

$$\implies \alpha - \epsilon < \mu - \beta$$

$$\implies \alpha - \gamma < \alpha - \epsilon < \mu - \beta$$

Here, $\alpha - \gamma$ is bidder k's surplus for P_A from separate selling of P_A and P_B, and $\mu - \beta$ is bidder k's surplus for P_B from the separate selling of P_A and P_B. Notice that since bidder k bid $\tau > \mu$, the second-highest bid for P_B is μ. Since bidder k pays μ, while his true value for P_B is β, his loss is equal to $\mu - \beta$. Thus, by misreporting his bid for P_B and forcing separate selling of P_A and P_B, bidder k's surplus is equal to

$$\beta - \mu + \alpha - \gamma < 0$$

Thus, by misreporting his bid for P_B and forcing separate selling of P_A and P_B, bidder k makes a surplus lower than his surplus from truthful bidding. As a result, truthful bidding is weakly dominant.

Thus, for Case 1(b), truthful bidding is weakly dominant when bidder k reports $\tau > \mu$.

Suppose bidder k bid $\tau = \mu$ (violating Assumption 3.1). In this case, there is a tie for P_B. If the ties are broken uniformly, then bidder k's surplus is

$$0.5 \times (\beta - \mu) + 0.5 \times 0 + \alpha - \gamma$$

which may be greater than zero. In this case, misreporting $\tau = \mu$ gives bidder k a higher surplus than zero. However, by Assumption 3.1 that the bidders' valuations come from continuous probability distributions functions, bidder k cannot exactly guess the highest bid for any bundle with non-zero probability. As a result, we discard the case where $\tau = \mu$.

In conclusion, for all of the cases laid out above, truthful bidding is weakly dominant. Thus, we show how truthful bidding is a weakly dominant strategy considering two bundles P_A and P_B.

From the discussion thus far, we observe that Case 1(b) is the non-trivial case. This case is characterized by the following features that hold true together. Let ϕ be the possible set of partitions of the item set L. For any partition P, $P \in \phi$ that the bidders can force the auctioneer to choose through misreporting, we have that

1. There is a bidder k such that he values some bundles $P^{h,k} \subseteq P$ the highest.

2. This bidder k does not value some bundles $P - P^{h,k}$ the highest.

3. Under truthful bidding, the auctioneer chooses a partition where the items in $Q_1 \subseteq P^{h,k}$ and items in any set Q_2, where $Q_2 \subseteq P - P^{h,k}$ are bundled together, and bidder k gets a surplus of zero.

4. If the items from $Q_1 \subseteq P^{h,k}$ and items from bundle $Q_2 \subseteq P - P^{h,k}$ are not

bundled together, then bidder k would get a positive surplus on Q_1 and zero surplus on Q_2.

To examine the incentives for truthful bidding, we extend the analysis discussed thus far by considering $P_A = Q_1,\ Q_1 \subseteq P^{h,k}$ and $P_B = Q_2,\ Q_2 \subseteq P - P^{h,k}$ for any partition $P, P \in \phi$ that the bidders can force force the auctioneer to choose through misreporting.

In the above analysis, if more than one bidder decide to misreport their bids by inflating them (i.e., reporting higher bids than their values for each item) , they stand to lose in a similar way, i.e., their surpluses are negative. Their gains from winning bundle $Q_1,\ Q_1 \subseteq P^{h,k}$ are offset by the losses from winning bundle $Q_2\ Q_2 \subseteq P - P^{h,k}$ for any partition $P,\ P \in \phi$ that the bidders can force through misreporting their bids.

In conclusion, truthful bidding at the item-level is a weakly dominant strategy under the condition that no bidder can guess any other bidder's bid for any item. $\square$

The intuition behind Proposition 3.1 is that for any bundle of items that gets sold in Step 3, the bundle is allocated to the bidder with the highest bid for the bundle, and the payment is the second-highest bid for the bundle. Therefore, if bidders do not report their values truthfully in step 1, their payments in Step 3 will be to their detriment. The reasoning is identical to the reasoning behind why truth-telling is weakly dominant in a second-price auction since the argument also holds for any bundle (whose valuation is the sum of the values of the items included in the bundle). Consequently, Proposition 3.1 implies that $\hat{u}_i^a = u_i^a$ for all $(i,a) \in BL$. In the Appendix, §3.11, I provide a detailed example illustrating why truthful bidding is a dominant strategy for the bidders.

3.5 Solving OBP

OBP is a bilevel, cubic, binary integer optimization problem with the inner optimization problem (3.17) being linked to the outer optimization problem through the variable λ. Given the computational complexity of OBP, we explore ways to simplify the problem structure. First, I explore ways to replace the complicating inner optimization problem with linear constraints, and then through relaxation and linearization techniques, we transform OBP to an equivalent linear integer programming formulation that can be solved efficiently using known techniques, such as the branch-and-bound based techniques. It is noteworthy that Proposition 3.1 allows us to equate $\hat{u}_i^a = u_i^a, (i,a) \in BL$ throughout the remainder of our analysis. As described next, in §3.5.1 I begin by replacing the inner optimization problem with a set of linear constraints.

3.5.1 Replacing the Inner Optimization Problem (3.17) with Linear Constraints

For brevity, I describe the technique to solve problem (3.17), when λ is fixed, in Appendix §3.9. Thus, if $\mathbf{x}$ is a solution to the inner optimization problem (3.17), then, for any feasible solution to OBP, it satisfies the following conditions (for a given λ vector.)

$$\sum_{i \in B} x_i^a = 1 \quad \forall a \in L, \tag{3.35}$$

$$z_i^{ab} \leqslant x_i^a \quad \forall i \in B, \quad \forall a \in L, b \in L, b \neq a, \tag{3.36}$$

$$z_i^{ab} \leqslant x_i^b \quad \forall i \in B, \quad \forall a \in L, b \in L, b \neq a, \tag{3.37}$$

$$z_i^{ab} \geqslant x_i^a + x_i^b - 1 \quad \forall i \in B, \forall a \in L, b \in L, b \neq a, \tag{3.38}$$

$$x_i^a \geqslant 0 \quad \forall (i,a) \in BL, \tag{3.39}$$

$$p^a - \sum_{\substack{b \in L \\ b \neq a}} \left[q1_i^{ab} + q2_i^{ba} - r_i^{ab} \right] \geqslant u_i^a - \sum_{\substack{b \in L, \\ b \neq a}} M^{ab} \lambda^{ab} \quad \forall (i,a) \in BL, \tag{3.40}$$

$$-M^{ab}\lambda^{ab} + q1_i^{ab} + q2_i^{ab} - r_i^{ab} \geqslant 0 \quad \forall i \in B \quad \forall a \in L, b \in L, b \neq a, \tag{3.41}$$

$$q_i^{ab,a} \geqslant 0, r_i^{ab} \geqslant 0 \quad \forall i \in B \quad \forall a \in L, b \in L, b \neq a \tag{3.42}$$

$$\sum_{(i,a)\in BL}\Big[u_i^a - \sum_{\substack{b\in L\\ b\neq a}} M^{ab}\lambda^{ab}\Big]x_i^a + \sum_{i\in B}\sum_{a\in L}\sum_{\substack{b\in L\\ b\neq a}} M^{ab}\lambda^{ab}z_i^{ab} = \sum_{i\in B}\sum_{a\in B}\sum_{\substack{b\in L\\ b\neq a}} r_i^{ab} + \sum_{a\in L} p^a. \quad (3.43)$$

This is because (3.17) can be solved as a linear program and constraints (3.35) – (3.43) represent the primal feasibility, dual feasibility, and strong duality conditions. However, note that these conditions are linear in the binary integer variables $\lambda^{ab}, a \in L, b \in L$. Moreover, the terms $-M^{ab}\lambda^{ab}x_i^a$ and $M^{ab}\lambda^{ab}z_i^{ab}$ cancel out for all $i \in B, a \in L, b \in L, b \neq a$ when $\mathbf{x}$ is optimal. This follows from Lemma 3.1. As a result, for a given λ vector (whose components are either 0 or 1), the strong duality condition (3.43) for an optimal solution $\mathbf{x}$, can be further simplified as,

$$\sum_{(i,a)\in BL} u_i^a x_i^a = \sum_{i\in B}\sum_{a\in B}\sum_{\substack{b\in L\\ b\neq a}} r_i^{ab} + \sum_{a\in L} p^a. \quad (3.44)$$

In Proposition 3.2, we formalize the argument that if $\lambda^{ab} \in \{0,1\}, \quad \forall \quad a, b \in L$ then (3.44) holds true given the inequalities (3.35) – (3.42).

Proposition 3.2 *If* $\lambda^{ab} \in \{0,1\}, \forall a, b \in L$, *then* (3.44) *holds true at optimality along with inequalities* (3.35) – (3.42).

Proof of Proposition 3.2. Let $\mathbf{x}$ be a solution to (3.35) – (3.42) and (3.43). If $\lambda^{ab} \in \{0,1\}, \forall \quad a, b \in L$, then by Lemma 3.1, we have that $\eta(\mathbf{x}) = 0$. As a result, (3.35) – (3.42) and (3.44) hold true. ▫

Next, I use Proposition 3.2 to transform OBP to a linear integer programming problem. The details are as discussed next in §3.5.2.

3.5.2 Relaxing and linearizing OBP

Here, I describe further steps to simply problem OBP. Besides replacing the inner optimization problem (3.17) with constraints (3.35) – (3.42), and (3.43), we make use of the following linearizations. To do so, I define variables $s_l^{ab,-i}$, $\phi_l^{ab,-i}$, $t_l^{ab,-i}$,

g_l^{ab}, and γ_l^{ab} as follows and impose the appropriate linear constraints.

$$s_l^{ab,-i} \equiv y_l^{a,-i} y_l^{b,-i}, \quad \phi_l^{ab,-i} \equiv \lambda^{ab} s_l^{ab,-i}, \quad t_l^{ab,-i} \equiv \lambda^{ab} y_l^{a,-i}, \quad g_l^{ab} \equiv \lambda^{ab} x_l^a, \quad \gamma_l^{ab} \equiv \lambda^{ab} z_l^{ab}. \tag{3.45}$$

For notational ease we define vectors $\mathbf{s}, \phi, \mathbf{t}, \mathbf{g}, \gamma$ corresponding to $s_l^{ab,-i}$, $\phi_l^{ab,-i}$, $t_l^{ab,-i}$, g_l^{ab}, and γ_l^{ab} respectively. This leads us to a linear integer programming problem, OBP-IP, where λ is the only binary integer decision variable. To avoid clutter and improve readability, we present the complete formulation of OBP-IP in Appendix §3.10. It is noteworthy that formulation OBP-IP includes the constraints (3.114) and (3.115). Proposition 3.3 below justifies the need for these constraints in OBP-IP.

Proposition 3.3 *The inclusion of constraints $g_i^{ab} = \gamma_i^{ab}$ for all $i \in B, a, b \in L$ and $t_l^{ab,-i} = \phi_l^{ab,-i}$ for all $i \in B, l \in B, l \neq i, a \in L, b \in L, b \neq a$ in the constraint set of OBP-IP ensures that the optimal objective value of OBP-IP is equal to the optimal objective value of OBP.*

Proof of Proposition 3.3.

The necessary conditions for the optimality of OBP is as discussed in §3.5.1. At optimality of OBP, it is a necessary condition that

$$\sum_{(i,a)\in BL} u_i^a x_i^a = \sum_{i\in B} \sum_{a\in B} \sum_{\substack{b\in L \\ b\neq a}} r_i^{ab} + \sum_{a\in L} p^a$$

with $x_i^a, (i,a) \in BL$ taking integral values. In the absence of constraints (3.114) and (3.115), there may exist fractional solutions to variables $x_i^a, (i,a) \in BL$ that satisfy constraint (3.98). Such fractional values of $x_i^a, (i,a) \in BL$ may lead to objective function values higher than that OBP-IP at optimality. Constraints (3.114) and (3.115) ensure that the necessary condition for optimality of OBP is satisfied. Since the optimal solution to OBP is integral (by definition), and the optimal solution to OBP-IP are integral (by Lemma 3.4), since the integral points in the feasible region of OBP-IP are identical to the feasible region of OBP, the optimal objective values of OBP-IP and OBP are identical. □

Finally, in Lemma 3.4, I show that only the variables λ need be constrained to take binary values, while all the remaining variables can be relaxed to take continuous values. Thus, the number of binary variables in OBP-IP is $\binom{|L|}{2} = \frac{|L|(|L|-1)}{2}$.

Lemma 3.4 *OBP-IP always has a feasible solution. The optimal solution to OBP-IP is such that* $\mathbf{y}^*, \mathbf{x}^*, \mathbf{z}^*, \mathbf{s}^*, \phi^*, \mathbf{t}^*, \mathbf{g}^*, \gamma^*$ *are integral, though they are allowed to take continuous values.*

Proof of Lemma 3.4. Consider any $a \in L$. Let $\lambda^{ab} \in \{0, 1\}$ for all $b \in L, b \neq a$ as required in OBP-IP. Then, the following values of the decision variables $\mathbf{p}, \mathbf{q}, \mathbf{r}, \pi$, of OBP-IP solve OBP-IP.

1. $r_i^{ab} = 0$ for all $i \in B$ and $b \in L$, and
2. $q1_i^{ab} + q2_i^{ab} = M_i^{ab}\lambda^{ab}$ for all $i \in B$ and $b \in L$.

Given vector λ, for each $a \in L$, define a set $L'(a)$ that is constructed as follows:

1. $L'(a)$ is empty initially,
2. $L'(a)$ is populated with elements $b \in L$ such that $\lambda^{ab} = 1$,
3. $L'(a)$ is further populated with elements $c \in L, c \notin L'(a)$ such that $\lambda^{bc} = 1$ or $\lambda^{bc} = 0$ for all $b \in L$ such that $\lambda^{ab} = 1$,
4. $L'(a)$ is populated with element a.

Intuitively, the set $L'(a)$ is the subset of set L that contains item a and all items bundled with a. The vector λ induces such a bundling (see definition 3.3.) For all items $c \in L'(a)$, set p^c such that $\sum_{c \in L'(a)} p^c = \sum_{c \in L'(a)} u^c_{i*(a)}$ where

$$i^*(a) = \arg\max_{k \in B}\{\sum_{c \in L'(a)} u_k^c\}. \tag{3.46}$$

In a similar manner, I define $L'(e)$ and $i^*(e)$ for all $e \in L, e \notin L'(a)$. We repeat this exercise for all item bundlings induced by λ. This way, we now have partition of set

L, and each item bundling in the partition can be defined by a unique item. Let $L_U \subseteq L$ be a set of all items that can be used to uniquely identify a bundling in the partition induced by λ. Thus, we have

$$\sum_{a \in L_U} \sum_{c \in L'(a)} p^c = \sum_{a \in L_U} \sum_{c \in L'(a)} u^c_{i*(a)}. \tag{3.47}$$

Notice that $\sum_{a \in L_U} \sum_{c \in L'(a)} u^c_{i*(a)}$ is a value I have obtained for the RHS of equation (3.98). Using this, I obtain values for $\mathbf{x}$ and $\mathbf{z}$ such that equation (3.98) is satisfied. Such values can be obtained as follows:

$$x^c_{i*(a)} = 1 \quad \forall \ \ a \in L'(a), \quad z^{cd}_{i*(a)} = 1 \quad \forall \ \ c, d \in L'(a), d \neq c. \tag{3.48}$$

$$x^c_j = 0 \quad \forall \ \ j \neq i^*(a), \qquad z^{cd}_j = 0 \quad \forall \ \ j \neq i^*(a), c, \ d \in L'(a), \ d \neq c.$$

With this selection of $\mathbf{x}$ and $\mathbf{z}$, we have that the LHS of (3.98) also equals

$$\sum_{a \in L_U} \sum_{c \in L'(a)} u^c_{i*(a)}.$$

With these values for decision variables in OBP-IP, it is easy to pick feasible values for the other decision variables of OBP-IP. Thus, we have a feasible solution for OBP-IP.

I now move onto the second statement of Lemma 3.4. For any vector λ feasible to OBP-IP, we have that $\mathbf{x}^*, \mathbf{z}^*$ are integer-valued at optimality of OBP-IP. This follows from the fact that problem SWLP has integral solutions, and that $\mathbf{x}^*, \mathbf{z}^*$ are a solution to the inner optimization problem in problem OBP, and that this inner optimization problem can be solved using a linear program with the structure of SWLP. The presence of the constraints (3.114) and (3.115) ensures that there are no terms involving variables $\mathbf{g}, \gamma, \mathbf{t}, \phi$ in the objective function of OBP-IP. Thus, they are not influenced by the objective function. This implies that since $\mathbf{x}^*, \mathbf{z}^*$ are integer-valued at optimality, we have that $\mathbf{g}^*$ and γ^* are integral at optimality. Variables $\mathbf{y}^*$ and $\mathbf{s}^*$ are integral at optimality, since, for a fixed vector λ, the variables $\mathbf{y}$ and $\mathbf{s}$ are not dependent on $\mathbf{x}$ and $\mathbf{z}$, and the structure of the problem involving $\mathbf{y}$ and $\mathbf{s}$ is identical to the problem SWLP. The variables $\mathbf{t}$ and ϕ do not feature in the objective function of OBP-IP because of constraint (3.115). Thus, they are not influenced by the objective function. This implies that since $\mathbf{y}^*, \mathbf{s}^*$ are integer-valued at optimality, we have that $\mathbf{t}^*$ and ϕ^* are integral at optimality. □

As we show in §3.6, Lemma 3.4 enables the development of an iterative algorithm using Benders decomposition approach to solve OBP-IP. Thus, the solution to OBP-IP will provide the auctioneer the optimal partition for Step 2 of the PBA (defined in §3.3) after the bids have been solicited in Step 1 (recall that the bids solicited in Step 1 are truthful by Proposition 3.1).

3.6 Benders Decomposition to Solve OBP-IP

The structure of OBP-IP allows for the application of the Benders decomposition technique as a solution algorithm. Consider a rewriting of problem OBP-IP where all terms involving λ are moved to the right-hand sides of the constraints where such terms are present. Essentially, problem OBP-IP can then be rewritten in the following compact notation that we call problem P-OBP-IP:

$$\textbf{P-OBP-IP:} \qquad \max \quad \mathbf{c}^T\mathbf{h} + \mathbf{0}^T\lambda, \tag{3.49}$$

$$\text{s.t.} \qquad \mathbf{A}\mathbf{h} \leqslant \mathbf{RHS}(\lambda), \tag{3.50}$$

$$\lambda^{ab} \in \{0, 1\} \quad \forall a, b \in L, b \neq a, \tag{3.51}$$

where $\mathbf{A}$ is the coefficient matrix, $\mathbf{h}$ represents the vector of all variables in OBP-IP other than λ, $\mathbf{0}$ is the zero-vector, and $\mathbf{RHS}(\lambda)$ represents the right-hand side vector of OBP-IP when all terms involving λ are moved to the right-hand side. It is readily observable that the variable λ represents the first-stage variables that are to be fixed for each iteration of Benders decomposition, while the variable $\mathbf{h}$ represents the second stage-variables whose values depend on the first stage variable λ. In the context of Benders decomposition, the problem P-OBP-IP can be decomposed into a master problem (MP), only involving binary integer variables λ, and a subproblem (SP) involving all other continuous variables in the relaxed linear program. In every iteration of the decomposition algorithm, the first stage variables in λ are first computed using MP, and then used to solve the second stage problem SP. At the end of each iteration, additional valid inequalities, generated after solving the SP, are added to the MP and the process is repeated until the algorithm converges to an optimal solution. Problems MP and SP are described below.

$$\textbf{MP:} \qquad \max \quad z, \tag{3.52}$$

$$\text{s.t.} \qquad \lambda^{ab} = \lambda^{ba} \quad \forall a, b \in L, b \neq a, \tag{3.53}$$

$$\lambda^{aa} = 0 \quad \forall a \in L, \tag{3.54}$$

$$\lambda^{ab} \in \{0, 1\} \quad \forall a, b \in L, b \neq a. \tag{3.55}$$

For a fixed value of $\lambda = \lambda^*$, the primal of the subproblem SP (to be defined shortly), which we call P-SP, is as follows.

$$\textbf{P-SP:} \qquad \max \quad \mathbf{c}^T\mathbf{h}, \tag{3.56}$$

$$\text{s.t.} \qquad \mathbf{Ah} \leqslant \mathbf{RHS}(\lambda^*), \tag{3.57}$$

$$\mathbf{h} \geqslant \mathbf{0}. \tag{3.58}$$

Note that P-SP is a linear program. This follows from Lemma 3.4. This is important because it allows us to have a dual formulation, which we call SP. The problem SP, when $\lambda = \lambda^*$, is as follows.

$$\textbf{SP:} \qquad \min \quad \mathbf{RHS}(\lambda^*)^T\mathbf{d}, \tag{3.59}$$

$$\text{s.t.} \qquad \mathbf{A}^T\mathbf{d} \leqslant \mathbf{c}, \tag{3.60}$$

$$\mathbf{d} \geqslant \mathbf{0}. \tag{3.61}$$

Next, I present the Benders decomposition algorithm in Algorithm 1. In Algorithm 1, SW_Max refers to the maximum social welfare obtainable from a feasible assignment of the items to the bidders. This is an upper bound on the maximum VCG revenue. The terms PBR and SSR refer to the VCG revenue obtained from pure bundling and separate selling respectively. They represent a (possibly tight) lower bound on the VCG revenue. PBR and SSR can be computed a priori from the values $u_i^a, \forall(i, a) \in BL$ provided in Step 1 of the PBA. The term ε is a tolerance parameter used as a stopping criterion for the algorithm.

Algorithm 1 Benders decomposition for OBP-IP.

1. Initialize $\lambda^* = \mathbf{0}$, receive values $u_i^a(i, a) \in BL$, set M
2. Initialize $UB = SW\_Max$, $LB = \max\{PBR, SSR\}$
3. **while** $UB - LB > \varepsilon$ **do**
    4. Solve problem SP using λ^* and obtain $\mathbf{d}^*$, which is the optimal value of $\mathbf{d}$ when SP is solved
    5. Set $UB = \min\{UB, \mathbf{RHS}(\lambda^*)^T\mathbf{d}^*\}$
    6. If $\mathbf{RHS}(\lambda^*)^T\mathbf{d}^*$ is unbounded, add constraint $\mathbf{RHS}(\lambda)^T\mathbf{d}^* \geqslant \mathbf{0}$ to MP
    7. Else add constraint $z \leqslant \mathbf{RHS}(\lambda)^T\mathbf{d}^*$ to MP
    8. Solve MP $\max z$ s.t. $\lambda^{ab} \in \{0, 1\}, \quad \forall a, b \in L, b \neq a$, and subject to the generated constraints in lines 6 and 7, and obtain λ^* and z^*, which are the optimal values of λ and z when MP is solved
    9. Set $LB = \max\{LB, z^*\}$

end

10. Return λ^*

As noted earlier, the MP is a mixed-integer program with binary variables λ and continuous variable z, while the problem SP is a linear program in $\mathbf{d}$. MP can be solved using standard techniques for mixed integer programming, while SP can be solved using standard techniques for linear programs. The cuts (optimality and feasibility) added to the master problem MP, in steps 6 and 7, are such that the objective of MP is (weakly) monotonically improving. The proof of this follows from standard literature on Benders decomposition (Bertsimas and Tsitsiklis 1997).

The construction of Algorithm 1 gives us an insight into the marginal value of bundling two items. Line 4 involves the computation of $\mathbf{d}^*$ that is used to update λ in line 8. The components of $\mathbf{d}^*$ can be used to compute the marginal change in revenue from changing a component of vector λ between two iterations. Thus, for the item bundling λ^* computed in the final iteration, the coefficient of λ^{ab*} in $z^* = \mathbf{RHS}(\lambda^*)^T(\mathbf{d}^*)$ in the final iteration is the marginal revenue benefit of putting together items a and b.

3.7 Numerical Experimentation

In this section, I present results from numerical experiments of OBP-IP. For every value of $|L|$ and $|B|$, as considered in Table 3.3a, Table 3.3b, and Table 3.4, I run 5 instances. For each of these 5 instances, a bidder's value for an item is an integer drawn from the set $\{1, 2, 3, \cdots, 100\}$ with equal probability. I set the value of M to 10,000 for all experiments. I run the computations with CPLEX 20.1 (via the DOCPLEX package on Python) on a MacBook Air 2015 on the macOS Catalina operating system with a 1.6 GHz Dual-Core Intel Core i5 processor, and a 4 GB 1600 MHz DDR3 RAM.

		$\|L\|$						
		3	4	5	6	7	8	9
$\|B\|$	3	16.82	23.11	26.98	25.79	28.88	26.30	27.75
	4	10.31	12.15	10.15	10.26	10.85	13.59	17.29
	5	3.76	5.67	5.35	4.99	4.43	4.13	5.01
	6	1.78	3.38	3.30	4.23	3.54	3.35	0.99
	7	2.72	3.22	3.07	3.62	2.92	1.77	0.93
	8	1.47	3.09	3.15	2.45	2.33	1.01	0.00
	9	0.09	0.64	0.48	0.34	0.74	0.38	0.09
	10	0.27	0.35	0.42	0.53	0.79	0.09	0.00
	11	1.96	0.56	0.20	0.24	0.31	0.00	0.14
	12	0.10	0.50	0.48	0.08	0.55	0.17	0.00

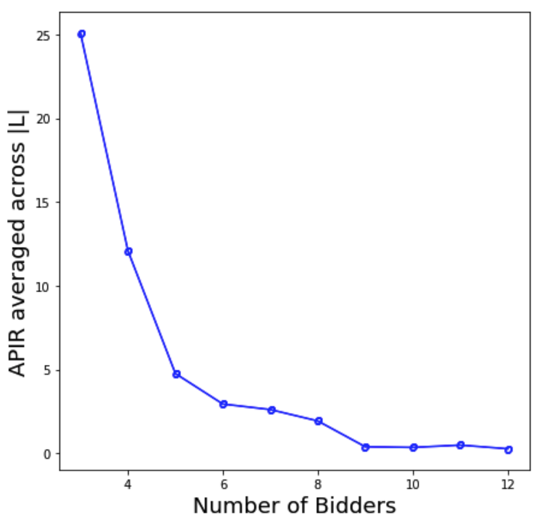
(a) APIR.

	$\|L\|$			
	3	4	5	6
3	0.034	0.157	0.595	1.656
4	0.045	0.207	0.492	12.543
5	0.086	0.317	1.440	10.439
6	0.104	0.365	1.460	20.083
7	0.140	0.529	2.906	25.353
8	0.146	0.615	5.928	58.539
9	0.203	0.872	5.121	77.946
10	0.232	1.050	8.217	90.942
11	0.319	1.288	8.635	99.118
12	0.335	1.402	11.687	168.214

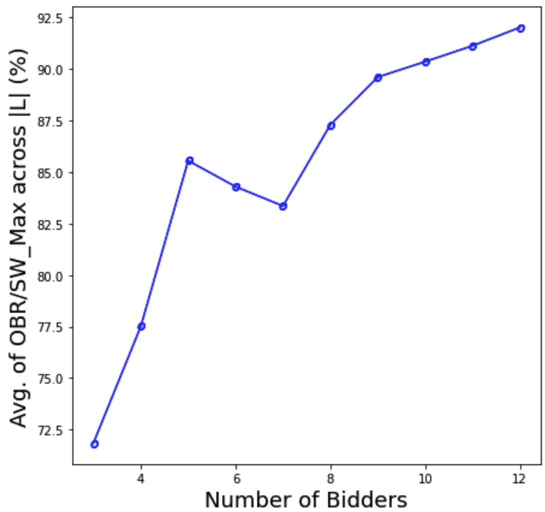
(b) Solution time (in secs.)

Table 3.3: APIR and solution times.

The results in Tables 3.3a, 3.3b, and 3.4 are obtained from the use of the default technique for integer programs available with CPLEX[6]. For each of the 5 instances for a value of $|B|$ and $|L|$, there is an associated percentage increase in revenue (PIR), defined as the percentage gain in revenue from optimal bundling over the revenue from separate selling. Table 3.3a reports the average PIR (APIR) across the 5 instances for a given value of $|L|$ and $|B|$. I observe a general trend that as $\frac{|L|}{|B|}$ decreases, APIR decreases. I attribute this to what we call the "competition effect" on revenue explained as follows: When $|B|$ increases, the revenue from separate selling increases, since higher competition implies higher bids in expectation. As a result, the increase in revenue provided by optimal bundling over the revenue from separate selling decreases. I also note a trend where APIR increases on the whole as $|L|$ increases when $|B|$ is fixed. I attribute this trend to the "item effect": More items on sale results in more possibilities for bundling and surplus extraction. However, the results show that this effect is not as pronounced as the "competition effect". The "competition effect" appears to dampen the "item effect" leading to low APIR despite increasing the number of items. Figure 3.2a depicts the variation of the average APIR (where the averaging is across $|L|$) with the number of bidders $|B|$. Note how the percentage increase in revenue decreases as a general trend when the number of bidders increases.



(a) APIR averaged across $|L|$ vs. $|B|$.



(b) $\frac{OBR}{SW\_Max}$ averaged across $|L|$ vs. $|B|$.

Figure 3.2: Average APIR and $\frac{\text{OBR}}{\text{SW\_MAX}}$ versus $|B|$.

Table 3.3b presents the average solution time for problem instances. I observe that the solution time increases substantially when $|L|$ is large. This is attributable to the fact that the number of components of vector λ increases as $|L|$ increases. The number of components of λ is upper-bounded by $\frac{|L|(|L|-1)}{2}$, and this bound can increase quickly as $|L|$ increases. As larger instances (i.e., instances where $|L| \geqslant 7$) take longer to run, we terminate the numerical runs after 10 minutes. Without a time limit constraint, I observe that computational times exceed 30 minutes. For instances where $|L| > 8$, I observe that computational times exceed an hour without any appreciable change to the optimality gap. To check the quality of the solutions, especially for instances where $|L| \geqslant 7$, whose computational time was restricted to 10 minutes, I examine the average of the ratio of bundling revenue to the maximum social welfare (the average is taken across all 5 instances for a given $|L|$ and $|B|$), since the maximum social welfare is the best achievable revenue for the auctioneer under any mechanism. These figures are shown in Table 3.4. When $|L| = 7, 8, 9$, OBR refers to the highest objective value of OBP-IP obtained at the time of termination of the computations (which was exactly after 10 minutes). For instances where $|L| = 3, 4, 5, 6$, OBR refers to the optimal bundling revenue. The term SW_Max refers to the maximum social welfare achievable in a problem instance. As $|B|$ increases, I note an increasing trend (on average) in the ratio of bundling revenue to the maximum social welfare. This is because higher competition eats into the surplus of every bidder, reducing the gap between the revenue from bundling and the maximum social welfare. Figure 3.2b depicts the variation of the average of the $\frac{\text{OBR}}{\text{SW\_Max}}$ ratio (averaged across the number of items) with the number of bidders $|B|$. Note the general increasing trend in the $\frac{\text{OBR}}{\text{SW\_Max}}$ ratio. An important observation from this analysis is that the benefits of item-bundling are best realized when the ratio of items to bidders is high, and that the benefits of item-bundling may not be very high when the auctioneer faces a large number of bidders. Since the "competition effect" dominates the effect of $|L|$ on auctioneer revenue when the number of bidders is high, the auctioneer may consider trading off between the costs of longer computation times and the revenue benefits of bundling. From the numerical results, it appears

		$\lvert L \rvert$						
		3	4	5	6	7	8	9
$\lvert B \rvert$	3	70.21	67.67	68.30	74.91	72.03	74.12	75.68
	4	76.55	73.97	77.08	74.84	82.03	77.52	80.59
	5	83.45	97.35	83.21	82.36	87.01	80.53	84.96
	6	85.85	79.35	86.65	83.87	86.32	83.43	84.54
	7	85.85	83.80	85.60	85.50	88.77	85.05	82.88
	8	87.31	83.89	81.20	90.22	90.23	90.39	87.78
	9	87.60	90.83	91.49	90.41	91.36	87.13	88.41
	10	91.01	88.71	90.81	91.60	90.64	89.60	90.07
	11	94.36	88.81	91.64	91.22	91.39	90.46	89.98
	12	91.92	91.01	92.64	91.63	90.97	93.24	92.63

Table 3.4: The $\frac{\text{OBR}}{\text{SW\_Max}}$ ratio (%).

that the auctioneer will not benefit considerably from the optimal bundle, and may choose to consider a suboptimal bundle in return for quick computation time, since the high competition would extract much of the surplus already.

My implementation of Benders decomposition is based on a functionality provided in CPLEX[7]. Given any mixed-integer linear program as an input, CPLEX uses the integer variables for the master-problem (MP), and uses the continuous variables for the sub-problem SP as a default setting if Benders decomposition is specified as a solution strategy. I use this CPLEX functionality because our decomposition algorithm is similar to the master- and sub-problem structure. I observe that the Benders decomposition algorithm (as implemented in CPLEX) is slower in convergence compared to the default algorithm to solve mixed-integer programs. Figure 3.3 shows the progression of the objective function value (of MP) with the iterations. Figure 3.3a shows an instance where we obtain the optimal solution (computation time: $4.426s$). Table 3.5 shows two problem instances using Benders decomposition where the algorithm did not terminate even after a large number of iterations. The column 'Opt. Gap' displays the reported optimality gap, and the column 'Solution Time' reports the time at which the algorithm was exogenously terminated. The column 'Objective' reports the highest objective value achieved at the time of termination. Figure 3.3b and Figure 3.3c show the progress of the objective value with the iterations of the Benders decomposition algorithm.

Instance	Objective	SW_Max	$\frac{\text{Objective}}{\text{SW\_Max}}$	Opt. gap	Solution time (secs.)
$\lvert L\rvert = 9, \lvert B\rvert = 3$	543	604	89.90%	11.23%	1216.82
$\lvert L\rvert = 12, \lvert B\rvert = 3$	765	1016	75.30%	32.81%	1221.29

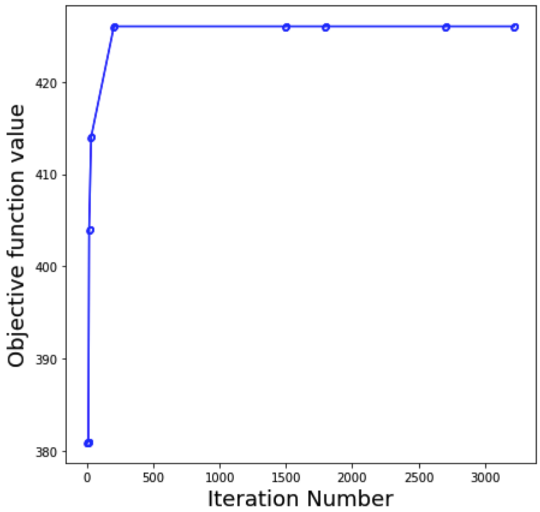
Table 3.5: Performance of Benders decomposition for two instances.

3.8 Conclusions

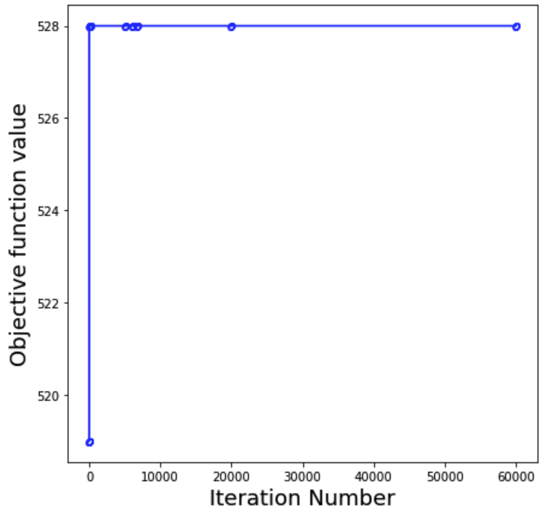
My problem is motivated as part of a broader problem on identifying revenue-improving mechanisms for multi-item settings (Sandholm et. al. 2004). As I showed in the illustration in §5.1, bundling can help improve the auctioneer's revenue. However, a challenge is that the auctioneer is not aware of the bidders' valuations of the items to make decisions on bundling. To solve this problem, I propose a 3-step mechanism, called the Pairwise Bundler Auction (PBA), which incentivizes the bidders to report their item-level valuations truthfully to the auctioneer *before* she bundles the items. The revenue from the PBA with the optimal bundling and the revenue from a VCG auction of the items using the same bundling are the same. As a result, the auctioneer no longer has to decide on an item-bundling based on *ex-ante* valuations to maximize her revenue (which, as noted, could be arbitrarily bad once the bidders' valuations are realized).

The main step of the PBA is Step 2, where the auctioneer optimally bundles the items using the bidders' valuations (reported truthfully in Step 1), to maximize her revenue from the VCG auction of the bundles in Step 3. In this regard, I formally describe the problem of computing the optimal bundling for revenue maximization under a VCG auction, followed by a formulation the optimal bundling problem. As part of problem formulation, I describe a series of simplification steps that reduced the problem of computing an optimal bundle to that of solving a binary integer linear program.

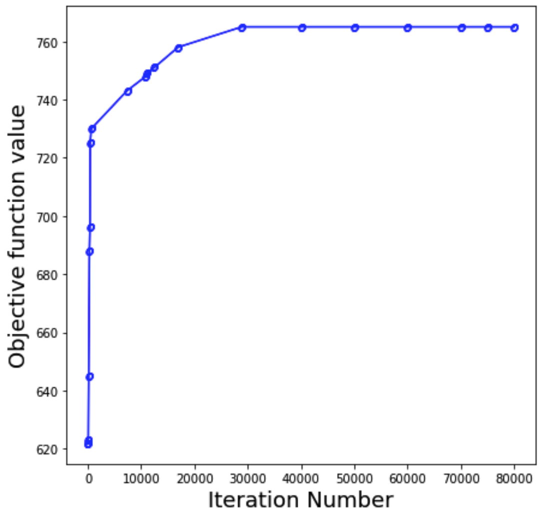
I show that the structure of the binary integer program used to solve the optimal bundling problem had the structure needed for the application of Benders decomposition. The binary integer variables are a part of the master problem, while the other variables are considered under the subproblem. Thus, the master problem



(a) $|L| = 6$ and $|B| = 3$.



(b) $|L| = 9$ and $|B| = 3$.



(c) $|L| = 12$ and $|B| = 3$.

Figure 3.3: Bundling revenue versus iterations.

is a binary integer program, while the subproblem is a linear program. The solution to the subproblem in the final iteration has an interesting interpretation: They represent the marginal benefit of having a pair of items together in the same bundle.

My results from numerical experiments show that the benefit of bundling over separate selling decreases as the number of bidders increases. I refer to this phenomenon as the "competition effect". I conclude that the benefits of bundling are best realized when the ratio of the number of items to the number of bidders per item is high. From our numerical experiments, we observe that the computation time of the optimal bundling problem is significantly high when the number of items is 7 or more. However, a feasible solution (bundling) that improves revenue over separate selling can be found fairly quickly by terminating the computations early. In addition to these results, we present a class of linearly-constrained binary quadratic optimization problems whose linear relaxations continue to yield binary solutions. This contributes to literature in operations research theory.

My work is promising for the following reasons. I identified a mechanism (the PBA) that incentivizes truthful bidding *before* the allocation rule is decided[8] when the bidders' bundle valuations are additive. To the best of our knowledge, there is no such precedent in literature. This has implications on auction design for revenue improvement in multi-item auctions. A more general line of work we have identified is auction design under a set of possible allocation rules, where bids (truthful or not) are solicited *before* the final allocation rule is decided from this set.

As part of future research, one may consider the problem of item bundling in settings with non-additive valuations. However, I point out that truthful bidding may not be the optimal strategy for bidders with non-addtive bundle valuations under the PBA sequence of events. Under this setting, perhaps other types of payments and allocation rules can be considered. Alternatively, the problem of identifying valuation functions for which truthful bidding is an optimal strategy may be examined.

Notes

[2]See https://yandex.com/support/direct/technologies-and-services/vcg-auction.html

[3]We use the terms “slots” and “items” interchangeably throughout the text.

[4]Essentially, a *bundling* refers to a *partition* of the set of items, and a *bundle* is a subset of items included in the partition, i.e., a bundle is assigned to exactly one bidder. I use the terms bundling and partitioning interchangeably.

[5]From here on, we refer to the online platform (i.e., the advertising service provider) as the auctioneer (pronoun: she) and the advertisers as bidders (pronoun: he).

[6]https://www.ibm.com/docs/en/icos/20.1.0?topic=optimization-solving-mixed-integer-programming-problems-mip

[7]https://www.ibm.com/docs/en/icos/20.1.0?topic=optimization-benders-algorithm

[8]The allocation of items to bidders depends on λ, which itself depends on the reported bids.

3.9 Appendix: Solving (3.22), (3.23), (3.24)

We examine the solution technique for (3.22) – (3.24). Here, (3.22) – (3.24) is structurally identical to problem (1) of Candogan et. al. (2015). As a result, its global optimum can be obtained through a linearization scheme applied on (3.22) – (3.24) (Candogan et. al. 2015). Candogan et. al. (2015) shows that solving the following linear program, SWLP, yields the solution to (3.22) – (3.24). Let $w_i^{ab} = M^{ab}\lambda^{ab}$ for all i and for all $a, b \in L$.

$$\textbf{SWLP:} \qquad \max \sum_{(i,a)\in BL} \Big[\hat{u}_i^a - \sum_{\substack{b\in L\\ b\neq a}} w_i^{ab}\Big] x_i^a + \sum_{i\in B}\sum_{a\in L}\sum_{\substack{b\in L\\ b\neq a}} w_i^{ab} z_i^{ab}, \tag{3.62}$$

$$\text{s.t.} \qquad \sum_{i\in B} x_i^a = 1 \quad \forall \;\; a \in L \quad \cdots (p^a), \tag{3.63}$$

$$z_i^{ab} \leqslant x_i^a \quad \forall \;\; i \in B, \quad \forall \;\; a \in L, b \in L, b \neq a \quad \cdots (q1_i^{ab}), \tag{3.64}$$

$$z_i^{ab} \leqslant x_i^b \quad \forall \;\; i \in B, \quad \forall \;\; a \in L, b \in L, b \neq a \quad \cdots (q2_i^{ab}), \tag{3.65}$$

$$z_i^{ab} \geqslant x_i^a + x_i^b - 1 \quad \forall \;\; i \in B, \forall \;\; a \in L, b \in L, b \neq a \quad \cdots (r_i^{ab}), \tag{3.66}$$

$$x_i^a \geqslant 0 \quad \forall \;\; (i,a) \in BL. \tag{3.67}$$

$$\textbf{DSWLP:} \qquad \min \sum_{i\in B}\sum_{a\in B}\sum_{\substack{b\in L\\ b\neq a}} r_i^{ab} + \sum_{a\in L} p^a, \tag{3.68}$$

$$\text{s.t.} \qquad -\sum_{\substack{b\in L\\ b\neq a}} \Big[q1_i^{ab} + q2_i^{ba} - r_i^{ab}\Big] \geqslant \hat{u}_i^a - \sum_{\substack{b\in L\\ b\neq a}} M_i^{ab}\lambda^{ab} \quad \forall \;\; (i,a) \in BL \quad \cdots (x_i^a), \tag{3.69}$$

$$-w_i^{ab} + q1_i^{ab} + q2_i^{ab} - r_i^{ab} \geqslant 0 \quad \forall \;\; i \in B \quad \forall \;\; a \in L, b \in L, b \neq a \quad \cdots (z_i^{ab}), \tag{3.70}$$

$$q1_i^{ab} \geqslant 0, r_i^{ab} \geqslant 0, q2_i^{ab} \geqslant 0 \quad \forall \;\; i \in B \quad \forall \;\; a \in L, b \in L, b \neq a. \tag{3.71}$$

Thus, if $\mathbf{x}$ solves problem SWLP, then the following conditions on $\mathbf{x}$ hold true.

Primal Feasibility Conditions:

$$\sum_{i\in B} x_i^a = 1 \quad \forall \;\; a \in L, \tag{3.72}$$

$$x_i^a \leqslant 1 \quad \forall \ \ (i,a) \in BL, \tag{3.73}$$

$$z_i^{ab} \leqslant x_i^a \quad \forall \ \ i \in B, \quad \forall \ \ a \in L, b \in L, b \neq a, \tag{3.74}$$

$$z_i^{ab} \leqslant x_i^b \quad \forall \ \ i \in B, \quad \forall \ \ a \in L, b \in L, b \neq a, \tag{3.75}$$

$$z_i^{ab} \geqslant x_i^a + x_i^b - 1 \quad \forall \ \ i \in B, \forall \ \ a \in L, b \in L, b \neq a, \tag{3.76}$$

$$x_i^a \geqslant 0 \quad \forall \ \ (i,a) \in BL. \tag{3.77}$$

Dual Feasibility Conditions:

$$p^a - \sum_{\substack{b \in L \\ b \neq a}} \left[q1_i^{ab} + q2_i^{ba} - r_i^{ab} \right] \geqslant \hat{u}_i^a - \sum_{\substack{b \in L \\ b \neq a}} w_i^{ab} \quad \forall \ \ (i,a) \in BL, \tag{3.78}$$

$$-w_i^{ab} + q1_i^{ab} + q2_i^{ab} - r_i^{ab} \geqslant 0 \quad \forall \ \ i \in B \quad \forall \ \ a \in L, b \in L, b \neq a, \tag{3.79}$$

$$q1_i^{ab} \geqslant 0, q2_i^{ab} \geqslant 0, r_i^{ab} \geqslant 0 \quad \forall \ \ i \in B \quad \forall \ \ a \in L, b \in L, b \neq a, \tag{3.80}$$

$$\pi_i^a \geqslant 0 \quad \forall \ \ (i,a) \in BL, \tag{3.81}$$

$$p^a \text{ is unrestricted} \quad \forall \ \ a \in L. \tag{3.82}$$

Strong Duality Conditions:

$$\sum_{(i,a) \in BL} \left[\hat{u}_i^a - \sum_{\substack{b \in L \\ b \neq a}} w_i^{ab} \right] x_i^a + \sum_{i \in B} \sum_{a \in L} \sum_{\substack{b \in L \\ b \neq a}} w_i^{ab} z_i^{ab} = \sum_{i \in B} \sum_{a \in B} \sum_{\substack{b \in L \\ b \neq a}} r_i^{ab} + \sum_{a \in L} p^a. \tag{3.83}$$

3.10 Appendix: Problem OBP-IP

$$\textbf{OBP-IP:} \qquad \max_{\boldsymbol{\lambda}, \mathbf{y}} \sum_{i \in B} \left[\sum_{\substack{l \in B \\ l \neq i}} \sum_{a \in L} \left[u_l^a y_l^{a,-i} - \sum_{\substack{b \in B \\ b \neq a}} M^{ab} t_l^{ab,-i} \right] + \sum_{\substack{l \in B \\ l \neq i}} \sum_{a \in L} \sum_{\substack{b \in L \\ b \neq a}} M^{ab} \phi_l^{ab} \right] \tag{3.84}$$

$$- \sum_{i \in B} \left[\sum_{l \in B} \sum_{a \in L} \left[u_l^{a,-i} x_l^a - \sum_{\substack{b \in L \\ b \neq a}} M_l^{ab,-i} g_l^{ab} \right] + \sum_{l \in B} \sum_{a \in L} \sum_{\substack{b \in L \\ b \neq a}} M_l^{ab,-i} \gamma_l^{ab} \right],$$

$$\text{s.t.} \qquad \sum_{\substack{l \in B \\ l \neq i}} y_l^{a,-i} = 1 \quad \forall \ \ a \in L \quad \forall \ \ i \in B, \tag{3.85}$$

$$y_l^{a,-i} \in \{0,1\} \quad \forall \;\; (l,a) \in BL, l \neq i, \quad \forall \;\; i \in B, \tag{3.86}$$

$$\lambda^{ab} = \lambda^{ba} \quad \forall \;\; a \in L, b \in L, b \neq a, \tag{3.87}$$

$$\lambda^{aa} = 0 \quad \forall \;\; a \in L, \tag{3.88}$$

$$\lambda^{ab} \in \{0,1\} \quad \forall \;\; a \in L, b \in L, b \neq a, \tag{3.89}$$

$$\sum_{i \in B} x_i^a = 1 \quad \forall \;\; a \in L, \tag{3.90}$$

$$z_i^{ab} \leqslant x_i^a \quad \forall \;\; i \in B, \quad \forall \;\; a \in L, b \in L, b \neq a, \tag{3.91}$$

$$z_i^{ab} \leqslant x_i^b \quad \forall \;\; i \in B, \quad \forall \;\; a \in L, b \in L, b \neq a \tag{3.92}$$

$$z_i^{ab} \geqslant x_i^a + x_i^b - 1 \quad \forall \;\; i \in B, \forall \;\; a \in L, b \in L, b \neq a, \tag{3.93}$$

$$x_i^a \geqslant 0 \quad \forall \;\; (i,a) \in BL, \tag{3.94}$$

$$p^a - \sum_{\substack{b \in L \\ b \neq a}} \left[q1_i^{ab} + q2_i^{ab} - r_i^{ab} \right] \geqslant u_i^a - \sum_{\substack{b \in L \\ b \neq a}} M^{ab} \lambda^{ab} \quad \forall \;\; (i,a) \in BL, \tag{3.95}$$

$$-M^{ab}\lambda^{ab} + q1_i^{ab} + q2_i^{ab} - r_i^{ab} \geqslant 0 \quad \forall \;\; i \in B \quad \forall \;\; a \in L, b \in L, b \neq a, \tag{3.96}$$

$$q1_i^{ab} \geqslant 0, q2_i^{ab} \geqslant 0, r_i^{ab} \geqslant 0 \quad \forall \;\; i \in B \quad \forall \;\; a \in L, b \in L, b \neq a, \tag{3.97}$$

$$\sum_{(i,a) \in BL} u_i^a x_i^a - \sum_{(i,a) \in BL} \sum_{\substack{b \in L \\ b \neq a}} M^{ab} g_i^{ab} + \sum_{(i,a) \in BL} \sum_{\substack{b \in L \\ b \neq a}} M^{ab} \gamma_i^{ab} = \sum_{i \in B} \sum_{a \in B} \sum_{\substack{b \in L \\ b \neq a}} r_i^{ab} + \sum_{a \in L} p^a, \tag{3.98}$$

$$s_l^{ab,-i} \leqslant y_l^{a,-i} \quad \forall \;\; l \in B, l \neq i, a \in L, b \in L, b \neq a, i \in B, \tag{3.99}$$

$$s_l^{ab,-i} \leqslant y_l^{b,-i} \quad \forall \;\; l \in B, l \neq i, a \in L, b \in L, b \neq a, i \in B, \tag{3.100}$$

$$s_l^{ab,-i} \geqslant y_l^{a,-i} + y_l^{b,-i} - 1 \quad \forall \;\; l \in B, l \neq i, a \in L, b \in L, b \neq a, i \in B, \tag{3.101}$$

$$\phi_l^{ab,-i} \leqslant \lambda^{ab} \quad \forall \;\; l \in B, l \neq i, a \in L, b \in L, b \neq a, i \in B, \tag{3.102}$$

$$\phi_l^{ab,-i} \leqslant s_l^{ab,-i} \quad \forall \;\; l \in B, l \neq i, a \in L, b \in L, b \neq a, i \in B, \tag{3.103}$$

$$\phi_l^{ab,-i} \geqslant \lambda^{ab} + s_l^{ab,-i} - 1 \quad \forall \;\; l \in B, l \neq i, a \in L, b \in L, b \neq a, i \in B, \tag{3.104}$$

$$t_l^{ab,-i} \leqslant \lambda^{ab} \quad \forall \;\; l \in B, l \neq i, a \in L, b \in L, b \neq a, i \in B, \tag{3.105}$$

$$t_l^{ab,-i} \leqslant y_l^{a,-i} \quad \forall \;\; l \in B, l \neq i, a \in L, b \in L, b \neq a, i \in B \tag{3.106}$$

$$t_l^{ab,-i} \geqslant \lambda^{ab} + y_l^{a,-i} - 1 \quad \forall \;\; l \in B, l \neq i, a \in L, b \in L, b \neq a, i \in B, \tag{3.107}$$

$$g_l^{ab} \leqslant \lambda^{ab} \quad \forall \;\; l \in B, a \in L, b \in L, b \neq a, \tag{3.108}$$

$$g_l^{ab} \leqslant x_l^{a} \quad \forall \;\; l \in B, a \in L, b \in L, b \neq a, \tag{3.109}$$

$$g_l^{ab} \geqslant \lambda^{ab} + x_l^{a} - 1 \quad \forall \;\; l \in B, a \in L, b \in L, b \neq a, \tag{3.110}$$

$$\gamma_l^{ab} \leqslant \lambda^{ab} \quad \forall \;\; l \in B, a \in L, b \in L, b \neq a, \tag{3.111}$$

$$\gamma_l^{ab} \leqslant z_l^{ab} \quad \forall \;\; l \in B, a \in L, b \in L, b \neq a, \tag{3.112}$$

$$\gamma_l^{ab} \geqslant \lambda^{ab} + z_l^{ab} - 1 \quad \forall \;\; l \in B, a \in L, b \in L, b \neq a, \tag{3.113}$$

$$g_l^{ab} = \gamma_l^{ab} \quad \forall \;\; l \in B, a \in L, b \in L, b \neq a, \tag{3.114}$$

$$t_l^{ab,-i} = \phi_l^{ab,-i} \quad \forall \;\; i \in B, l \in B, l \neq i, a \in L, b \in L, b \neq a, \tag{3.115}$$

$$\text{All variables except } p^a, a \in L \text{ are non-negative.} \tag{3.116}$$

3.11 Appendix: Illustration of Truthful Bidding

Let $L = \{1, 2\}$ and let $B = \{1, 2\}$. Let the true valuations for the items be as follows: $u_1^1 = 1, u_1^2 = 5, u_2^1 = 2, u_2^2 = 3$.

	1	2
1	1	5
2	2	3

Table 3.6: True Valuations.

Instance 1 (Overreporting). Let $\hat{u}_1^1 = 3 > u_1^1 = 1, \hat{u}_1^2 = u_1^2 = 5, \hat{u}_2^1 = u_2^1 = 2, \hat{u}_2^2 = u_2^2 = 3$. Note that bidder 1 has misreported his value for item 1 at 3 instead of its true value 1.

If the two items were bundled, Bidder 1 wins item 1 and item 2 since $\hat{u}_1^1 + \hat{u}_1^2 > \hat{u}_2^1 + \hat{u}_2^2$ and pays an amount equal to $\hat{u}_2^1 + \hat{u}_2^2 = 2 + 3 = 5$. Bidder 1's payoff is

	1	2
1	$\underline{(3)}$ 1	5
2	2	3

Table 3.7: Misreported valuation underlined in parentheses with the true valuation outside the parentheses.

$u_1^1 + u_1^2 - (u_2^1 + u_2^2) = 1 + 5 - (2 + 3) = 1$. If bidder 1 had reported truthfully, i.e., $\hat{u}_1^1 = u_1^2$, then he would win item 1 and item 2 since $\hat{u}_1^1 + \hat{u}_1^2 > \hat{u}_2^1 + \hat{u}_2^2$ and pays an amount equal to $\hat{u}_2^1 + \hat{u}_2^2 = 2+3 = 5$. His payoff then is payoff is $u_1^1 + u_1^2 - (u_2^1 + u_2^2) = 1 + 5 - (2 + 3) = 1$.

If the two items were sold separately, Bidder 1 would win item 1 and item 2 since $\hat{u}_1^1 > \hat{u}_2^1$ and $\hat{u}_1^2 > \hat{u}_2^2$. Bidder 1 would pay $\hat{u}_2^1 = 2$ for item 1 and $\hat{u}_2^2 = 3$ for item 2. His payoff would be $u_1^1 - \hat{u}_2^1 + u_1^2 - \hat{u}_2^2 = 1 - 2 + 5 - 3 = 1$. If bidder 1 had reported truthfully, i.e., $\hat{u}_1^1 = u_1^2$, then bidder 1 loses item 1 and pays zero for item 1. Bidder 1 wins item 2 since $\hat{u}_1^2 > \hat{u}_2^2$ and pays $\hat{u}_2^2$ for item 2. His payoff from reporting truthfully is $u_1^2 - \hat{u}_2^2 = 5 - 3 = 2 > 1$. This different of 1 unit of surplus is because bidder 1 did not have to lose 1 unit of surplus (from acquiring item 1) under truthful bidding that he lost under misreporting his value for item 1. Note that bidder 1 continues to make the same surplus of $5 - 3 = 2$ units from acquiring item 2 under misreporting that he did even under truthful bidding. Thus, the loss of 1 unit of surplus is not compensated for by surplus gains on item 2.

Instance 2 (Underreporting). Let $\hat{u}_1^1 = 0.5 < u_1^1 = 1, \hat{u}_1^2 = u_1^2 = 5, \hat{u}_2^1 = u_2^1 = 2, \hat{u}_2^2 = u_2^2 = 3$. Note that bidder 1 has misreported his value for item 1 at 0.5 instead of its true value 1.

	1	2
1	$\underline{(0.5)}$ 1	5
2	2	3

Table 3.8: Misreported valuation underlined in parentheses with the true valuation outside the parentheses.

If the two items were bundled, bidder 1 would win item 1 and item 2, and would pay $\hat{u}_2^1 + \hat{u}_2^2 = 2+3 = 5$. His surplus would be $u_1^1 + u_1^2 - (\hat{u}_2^1 + \hat{u}_2^2) = 1+5-(2+3) = 1$. If bidder 1 had reported truthfully, i.e., $\hat{u}_1^1 = u_1^1$, he would win item 1 and item 2, and would pay $\hat{u}_2^1 + \hat{u}_2^2 = 2 + 3 = 5$. His surplus would be $u_1^1 + u_1^2 - (\hat{u}_2^1 + \hat{u}_2^2) = 1 + 5 - (2 + 3) = 1$. Thus, misreporting his bid is not providing bidder 1 with any additional surplus gain.

If the two items were sold separately, then bidder 1 would win item 2 and bidder 2 would win item 1. Bidder 1 would pay $\hat{u}_2^2$ for item 2 and make a surplus of $u_1^2 - \hat{u}_2^2 = 5 - 3 = 2$. If bidder 1 would have reported truthfully, then bidder 1 would win item 2 and bidder 2 would win item 1. Bidder 1 would pay $\hat{u}_2^2$ for item 2 and make a surplus of $u_1^2 - \hat{u}_2^2 = 5 - 3 = 2$. Thus, misreporting his bid is not providing bidder 1 with any surplus gain.

Instance 3 (Overreporting). Let $\hat{u}_1^1 = u_1^1 = 1, \hat{u}_1^2 = 7 > u_1^2 = 5, \hat{u}_2^1 = u_2^1 = 2, \hat{u}_2^2 = u_2^2 = 3$. Note that bidder 1 has misreported his value for item 2 as 7 instead of the true value 5.

	1	2
1	1	$\underline{(7)}$ 5
2	2	3

Table 3.9: Misreported valuation underlined in parentheses with the true valuation outside the parentheses.

If the two items were bundled, bidder 1 would win both item 1 and item 2. His surplus would then be $u_1^1 + u_1^2 - (\hat{u}_2^1 + \hat{u}_2^2) = 1 + 5 - (2 + 3) = 1$. If bidder 1 bid truthfully, i.e., $\hat{u}_1^2 = u_1^2$, then bidder 1 would win item 1 and item 2 with a surplus of $u_1^1 + u_1^2 - (\hat{u}_2^1 + \hat{u}_2^2) = 1 + 5 - (2 + 3) = 1$.

If the two items were sold separately, then bidder 2 would win item 1 since $\hat{u}_2^1 > \hat{u}_1^1$ and bidder 1 would win item 2 since $\hat{u}_1^2 > \hat{u}_2^2$. Bidder 1 would pay $\hat{u}_2^2 = 3$ for item 2. Bidder 1's surplus would be $u_1^2 - \hat{u}_2^2 = 5 - 3 = 2$. If bidder 1 had reported his value truthfully, i.e., $\hat{u}_1^2 = u_1^2$, then he would have won item 2 and lost item 1. Then, his surplus would be $u_1^2 - \hat{u}_2^2 = 5 - 3 = 2$.

Instance 4 (Underreporting). Let $\hat{u}_1^1 = u_1^1 = 1, \hat{u}_1^2 = 3.5 < u_1^2 = 5, \hat{u}_2^1 = u_2^1 = 2, \hat{u}_2^2 = u_2^2 = 3$ Note that bidder 1 has misreported his value for item 2 as 3.5 instead of the true value 5.

If the two items were bundled together, then say bidder 2 would win items 1

	1	2
1	1	$(\underline{3.5})$ 5
2	2	3

Table 3.10: Misreported valuation underlined in parentheses with the true valuation outside the parentheses.

and 2 since $\hat{u}_2^1 + \hat{u}_2^2 > \hat{u}_1^1 + \hat{u}_1^2$. Bidder 1's surplus would be zero. If bidder 1 bid truthfully, then bidder 1 would win items 1 and 2 and would make a surplus of $u_1^1 + u_1^2 - (\hat{u}_2^1 + \hat{u}_2^2) = 5 + 1 - (2 + 3) = 1$. Thus, bidder 1 gains from bidding truthfully.

If the two items were sold separately, then bidder 1 would win item 2 and would pay $\hat{u}_2^2 = 3$. His surplus would be $u_1^2 - \hat{u}_2^2 = 5 - 3 = 2$. If bidder 1 bid truthfully, then bidder 1 would win item 2 and would pay $\hat{u}_2^2 = 3$. His surplus would be $u_1^2 - \hat{u}_2^2 = 5 - 3 = 2$.

We note that in all four instances, a bidder would weakly gain by bidding truthfully, whether or not the items are bundled. This demonstrates that bidding truthfully is a weakly dominant strategy. Similar demonstrations can be obtained in cases with more than 2 items and 2 bidders by assuming that items 1 and 2 are bundles and by assuming that bidder 2's valuation is the second-highest of item 1 and item 2 respectively (when considering that item 1 and 2 are being sold separately) and by assuming that bidder 2's valuation for items 1 and 2 together is the second-highest of all bidders for items 1 and 2 together (when considering that item 1 and 2 are being sold as a bundle).

Chapter 4

Optimizing Offer Sets for Multi-Item Simultaneous Auctions

4.1 Introduction

Simultaneous sealed-bid auctions are a commonly-used format for the sales of several items simultaneously to multiple interested buyers (henceforth referred to as bidders). In a simultaneous sealed-bid auction, the auctioneer auctions off several items concurrently, with bidders submitting sealed bids for the items they are interested in separately. The use of simultaneous sealed-bid auctions can be seen in the US oil and natural gas auctions, timber auctions, Essential Air Service auctions, and the Regional Connectivity Scheme auctions (Hendricks et. al. 2014, Athey and Levin 2001, Essential Air Service 2018, RCS Scheme 2016) and eBay auctions (Overby and Kannan 2015).

In practice, bidders have to ascertain the values of the items they are interested in bidding for before they submit their bids for the items (Golrezaei and Nazerzadeh 2017, Ye 2007, Hendricks et. al. 2014, Athey and Levin 2001). For example, when bidders have to bid for oil tracts, they conduct inspections to obtain estimates of the various tracts put up for sale. This process of conducting due diligence, or simply, an inspection, is a costly affair in practice. An analogous phenomenon can also be found in online auctions on websites such as eBay. Here, bidders need to identify items that they would be interested in, and they incur search costs in doing so. These search costs determine the number of auctions a bidder can participate in (Overby and Kannan 2015).

The high costs of inspection result in the following consequences for the bidders and the auctioneer: (a) bidders are limited in the number of items they can submit bids for, since bid formulation is preceded by costly inspection, and (b) bidders may try to avoid inspecting items they believe the rival bidders are inspecting so that they avoid competition at the bidding stage (Haile et. al. 2010, Hendricks et. al. 2014). Point (a) refers to the fact that the bidders have limited inspection budgets. As a result, bidders only inspect (and bid for) as many items as their inspection budgets permit. Point (b) refers to the fact that bidders may try to avoid other bidders, and this affects the number of bidders on each item that is put up for sale. In short, (a) and (b) influence a bidder's participation decisions in the simultaneous auction of the various items put up for sale.

In this chapter, we examine the auctioneer's problem of identifying a subset of items (out of all available items) to put up for sale under a simultaneous second-price auction, given points (a) and (b) that influence the bidders' participation decisions. The auctioneer's search for this subset is driven by the fact that she cannot accurately predict the participation decisions of the bidders on a set of items she chooses to put up for sale. That is, the auctioneer does not have knowledge of either the bidders' inspection budgets or their participation decisions until *after* they have submitted their bids, and she needs to decide on a set of items to put up for sale *before* they do so. We show an illustration below to describe how the bidders' participation decisions can impact the auctioneer's expected revenue.

Consider a simple instance with 9 identical bidders looking to acquire one item each from a set $I = \{1, 2, 3\}$ of items. Let the bidders' private values for item 1 be distributed uniformly between $[1, 2]$, item 2 be distributed uniformly between $[2, 3]$, and item 3 be distributed uniformly between $[3, 4]$. In such a setting, the auctioneer's expected revenue from the sale of items in set I is maximized when the bidders participate in a manner where there are 3 bidders on each one of the items. When this happens, the auctioneer's expected revenue from a simultaneous second-price auction is $1+\frac{2}{4}+2+\frac{2}{4}+3+\frac{2}{4} = 7.5$[9]. However, if it were such that there were 4 bidders on item 2 and 5 bidders on item 3, then the auctioneer's expected revenue would be $2 + \frac{3}{5} + 3 + \frac{4}{6} = 6.2667 < 4.5$. The worst-case expected revenue is realized when all 9 bidders bid for item 1. In this case, the auctioneer's expected revenue is $1 + \frac{8}{10} = 1.8 < 6.2667 < 7.5$. This simple illustration demonstrates how the uncertainty in the bidders' participation decisions can make a difference to the auctioneer's expected revenue for the same set I of items put up for sale.

We will, henceforth, refer to the various possible sets of participation decisions of the bidders as the bidders' *participation patterns.* For any set of items the auctioneer puts up for sale, there are several possible participation patterns. For example, if the auctioneer puts up a set Z of items for sale, she could potentially see all of the bidders bidding for just one item in the set Z with all others remaining unsold, or she could potentially see bidders participating in large numbers for some items in Z but very thinly on some others. The total expected revenue from putting up item set Z for sale depends on what participation pattern is realized, and this participation pattern is uncertain.

The auctioneer needs to identify a subset of items to put up for sale keeping in mind the uncertainty in participation patterns of the bidders. We make use of a minimax regret criterion to solve the auctioneer's problem of identifying a set of items to put up for sale. We call such a set the *offer set.* The uncertainty in the problem arises from the fact that the auctioneer does not know what the bidders' participation pattern will look like for any offer set. In the light of this uncertainty, a minimax regret criterion would be appropriate, as opposed to an expected expected[10] revenue maximization criterion. This is because the expected expected revenue maximization criterion makes use of the bidders' mixed strategy Nash equilibrium probability distributions, and these distributions are difficult to estimate in practice. In contrast, the minimax regret criterion is a distribution-free criterion, and will thus be well-suited to our problem. To the best of our knowledge, there has never been a precedent for an optimal subset selection problem of this nature in management literature.

In the context of our problem, the term *regret* alludes to the "cost of uncertainty" (borne by the auctioneer) posed by the uncertainty in the bidders' participation decisions. As we showed in the illustration above, the uncertainty in the bidders' participation decisions (or participation patterns) impacts the auctioneer's expected revenue. In practice, it may not be possible for the auctioneer to estimate the probability distributions of each bidder's participation decisions. Therefore, an approach that aims to minimize the worst-case "cost of uncertainty" is appropriate. We also briefly discuss why two other common approaches to dealing with uncertainty, i.e., the worst-case approach and expected value maximization approach, are not appropriate here.

Our key result is that the problem of identifying the max-regret minimizing subset of items is achievable in polynomial time when (i) the bidders' item valuations are additive, (ii) bidders' valuations for an item are independently and identically distributed, and (iii) the size of the state space modeling uncertainty is polynomially bounded.

A modeling construct we make use of in our problem is that for any set of items put up for sale, there is a discrete set of possible expected revenue amounts the auctioneer could gain. Each value of expected revenue corresponds to a participation pattern, and the number of possible participation patterns is finite in number. As a result, we are able to rank-order the possible values of expected revenue and treat each value of expected revenue as an outcome of a state of nature.

Our results are insightful in business settings that involve the simultaneous auctions of multiple items where the bidders' participation is limited to a subset of the entire inventory of items either because item inspection is costly or because search costs can be very high. A known example of costly pre-bidding item inspection can be found in the context of auctions for natural resource tracts. Here, bidders do not bid for every tract that is open for auction (Haile et. al. 2010, Hendricks et. al. 2014). Additionally, bidding data described in Haile et. al. (2010) and Hendricks et. al. (2014) shows that the number of bids (and bid values) per tract is, quite often, a low number. Hendricks et. al. (2014) attribute this to the large number of oil tracts being made available in every auction. We believe that targeted sales of the oil tracts (as opposed to opening up all of them for sale) could mitigate the problem of low competition and low revenue. This chapter provides a first step towards understanding the problem of optimally deciding on an offer set of tracts. Along similar lines, the eBay auction context aptly describes a setting where bidder participation in auctions is limited by the bidders' search costs (Overby and Kannan 2015). In either setting, it is worthwhile for the seller to determine an optimal set of items to put up for sale considering the uncertainty in the bidders' participation patterns.

In §4.2 we review the relevant literature and explain our contributions. In §4.3, we describe the model preliminaries and formally state our problem. In §4.4, we describe the results and present their proofs. In §4.5, we examine the problem of minimax regret when we only consider a state space that is polynomially bounded and review the illustration in §4.1. In §4.6, we conclude the chapter and describe some potential directions for future research.

4.2 Literature Review

Our work contributes to literature in auctions involving pre-bidding inspection costs or search costs and the broader literature on auctions involving the sale of multiple items.

Literature on costly pre-bidding inspection. Auctions with costly information acquisition before bidding are well-studied in literature. Hendricks et. al. (2014) describe how oil firms need to invest in costly information acquisition on the oil tracts before formulating their bids. Athey and Levin (2001) describe how bidders conduct a costly information acquisition process before placing their bids in the context of timber auctions. Ye (2007) and Golrezaei and Nazerzadeh (2017) study a two-stage auction format that involves costly information acquisition after a first stage of non-binding bidding. A similar notion of costly participation can be seen in the context of internet auctions. Such participation costs are known as *search costs* (Overby and Kannan 2015). Although search costs are generally understood from the lens of posted-price settings, the idea extends to settings involving auctions. The key difference is that in auctions, the prices are set based on the number of participants in the auctions, and search costs affect bidder participation (Overby and Kannan 2015).

Literature on mechanisms involving the sale of multiple items. Auctions involving multiple items are conducted using simultaneous sales, where the items are auctioned off at the same time, or sequential sales, where the items are auctioned off one after the other. One of the earliest papers on multi-item auctions is Engelbrecht-Wiggans and Weber (1979) where the authors discuss equilibrium behaviour in simple multi-item concurrent auctions. Rosenthal and Wang (1996) consider a multi-item setting and discuss equilibrium bidding behaviour involving common values and bid synergies. We refer the readers to de Vries and Vohra (2003) and Pekec and Rothkopf (2003)) for a comprehensive survey on the related area of combinatorial auction design, where bidders may have synergies in the values of the items, i.e., a bidder may value a bundle at an amount different from the sum of his values for the individual items that make up the bundle. Combinatorial auctions allow for the expression of such synergies.

Literature on congestion games and potential games. Collective information acquisition decisions (or participation decisions) give rise to games that are similar to congestion games (Rosenthal 1973). Agents choose items to acquire information on based on how many other agents choose the same items to acquire information on, and would prefer the items that would give them the highest expected payoff given the strategies of the other players. In potential games, the collective decisions of the players can be mapped onto a quantity known as the potential function (Monderer and Shapley 1996). In this work, the collective participation decisions of the bidders can be mapped onto the auctioneer's expected revenue function. The idea of the participation decisions of the bidders impacting the auctioneer's revenue has been discussed in Overby and Kannan (2015).

Literature on the sale of assets. Our work is related to the asset-selling problem well-studied in management literature. In the asset-selling problem, the decision-maker decides on an optimal proportion of her assets to sell away. In its most basic form, the asset selling problem was studied by Karlin (1962). Other pieces of literature include those of Prastacos (1983) and Ahn et. al. (2021). A common assumption in much of literature is that the asset is divisible, making it possible to sell off "fractions" of the asset. A fundamental manner in which our work differs from those of literature is the discrete nature of the assets the seller has in our context. Our problem examines a setting where the items on sale are indivisible, i.e., each item must be sold as a whole or not sold at all.

Our contribution to literature. We highlight the fact that there exists uncertainty in the bidders' participation decisions that we call the *participation patterns.* As we showed in the illustration in §4.1, the same set of items put up for auction can result in differing values of expected revenue depending on what participation pattern plays out. Putting up a certain set of items on auction can give the auctioneer a very high best-case expected revenue but also a very low worst-case expected revenue. Therefore, the auctioneer needs to prudently choose a subset of items (out of the entire inventory of items) to put up for sale. To the best of our knowledge, there is no paper in literature that has examined a question of this nature. In this regard, we use a minimax regret criterion to identify an optimal subset for putting up for sale. Our work is related to literature on the sale of assets with two important distinctions from literature: (i) we consider sales where the revenue is determined through an auction and (ii) the assets are heterogeneous and indivisible. Thus, we hope to have contributed to literature on asset-selling by opening up a new avenue for research that, to the best of our knowledge, has not been studied before.

4.3 Model

4.3.1 Preliminaries

Setting and assumptions. There is a set I of heterogeneous and indivisible items. There is a set of bidders B. The auctioneer is looking to sell some or all of the items in the set I to the bidders in set B. Bidders in the set B are looking to purchase some or all of the items the auctioneer chooses to put up for sale. The auctioneer uses a simultaneous second-price sealed bid auction to sell the items she chooses to put up for sale.

Suppose the auctioneer chooses to put up a set $Z \subseteq I$ for sale. The bidders choose a set of items from set Z to inspect and learn their private value for the items they choose to inspect. Before inspection, each bidder $b \in B$ has a private value for item i, denoted by random variable X_{ib}, where X_{ib} is distributed with a PDF f_i and CDF F_i with finite support $[l_i, u_i]$. We also assume that the CDF $F_i,\ i \in I$ are all regular, i.e., the hazard rate of $F_i,\ i \in I$ increases. Each bidder $b \in B$, needs to inspect item i before submitting a bid for item i. Thus, for a given $i \in I$, the random variables $X_{ib},\ b \in B$ are independent and identically distributed (IID). A bidder does not participate in the auction for an item if he does not inspect it first. After inspection, bidder b knows his private value for item i as x_{ib}, and participates in the auction for item i by submitting a sealed bid for item i. Bidder $b \in B$ values item set $J \subseteq I$ as $\sum_{i \in J} x_{ib}$. If he wins item i, he pays the second-highest bid for item i or the reserve price r_i for item i. We assume that $r_i = l_i$ for all $i \in I$. Note that this may not be the optimal reserve price for the auctioneer. However, we do not go into the problem of determining an optimal reserve price for the items in our analysis. This timeline of events is summarized in Figure 4.1.

Each bidder $b \in B$ has inspection budget n_b, where n_b is an integer. We make

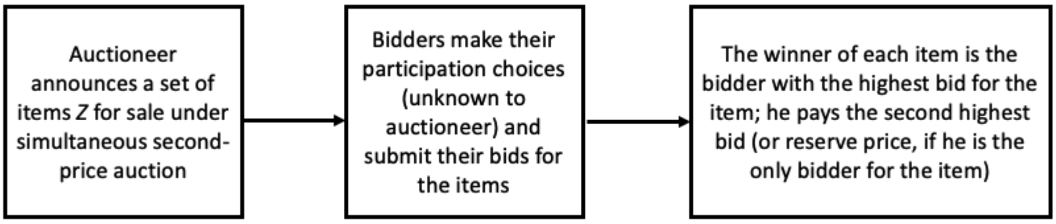


Figure 4.1: Timeline of events.

the following assumptions:

(i) It costs 1 unit to conduct an inspection of any item,

(ii) $n_b \in \{N_{bL}, N_{bL}+1, \cdots, N_{bU}-1, N_{bU}\}$ where N_{bL} and N_{bU} are integers, (iii) a bidder does not compete against himself,

(iv) a bidder will attempt to acquire as many items as possible under the simultaneous second-price auction. In this regard, each bidder will inspect as many items as he can (and bid for them), based on what n_b is and what the cardinality of the offer set is. Thus, each bidder b will spend all of his inspection budget n_b.

(v) n_b is known to bidder B and is unknown to the auctioneer and unknown to the other bidders,

(vi) the fact that $n_b \in \{N_{bL}, N_{bL}+1, \cdots, N_{bU}-1, N_{bU}\}$ is known to the auctioneer and the other bidders, and

(vii) $N_{bL} = 1$ for all $b \in B$.

The bidder's objective is to improve his surplus by acquiring as many items as possible in the set of items offered by the auctioneer under a second-price auction. We assume that the bidders' degree of uncertainty about the valuation of the items is high enough to necessarily require an inspection before bid submission. This is often the case in practice, for example, in oil and natural gas auctions (Hendricks et. al. 2014) and influences bidder participation decisions as noted in the observations of Hendricks et. al. (2014) and Haile et. al. (2010).

Modeling uncertainty in participation pattern. We describe the nature of the uncertainty as follows. For any set Z of items the auctioneer puts up for sale, the auctioneer can see a participation pattern that

1. gives her the highest expected revenue from the sale of items in set Z or,

2. gives her the second-highest expected revenue from the sale of items in set Z or,

3. gives her the third-highest expected revenue from the sale of items in set Z or,

4. gives her the fourth-highest expected revenue from the sale of items in set Z $\cdots\cdots$ or,

5. gives her the worst expected revenue from the sale of items in set Z

Let $p \in \{H, 2, 3, 4, \cdots, L\}$ be the parameter that represents the state of the nature, i.e., the bidder participation pattern, that is realized when the auctioneer puts up any set Z of items for sale. For example, if $I = \{1, 2, 3, 4, 5\}$ and $Z = \{2, 3, 4\}$, then $p = H$ represents a participation pattern on items in Z where the auctioneer earns the highest expected revenue from the sale of the items in Z, $p = 3$ represents a participation pattern on items in Z where the auctioneer earns the third-highest expected revenue from the sale of the items in Z. It is possible that two states p_1 and p_2 where, without loss of generality, $p_1 = p_2 + 1$ may share the same expected revenue value. Note that for a set Z of items put up for sale and some state p that materializes, it may be possible that some items in set Z do not receive any bids. Additionally, note that the bidders' inspection budgets n_b, $b \in B$ also influence the participation patterns.

Our motivation for this modeling construct is based on the fact that (i) only a discrete number of states (or participation patterns) is possible, since there is a countably finite number of possible participation patterns when set Z is put up for sale, and that (ii) there is an expected revenue associated with each participation pattern, making it possible to rank-order the expected revenue values across participation patterns.

Without loss of generality, we could consider the number of states for every set $Z \subseteq I$ to be the same. To do this, for every set of items Z offered, we can set the number of possible states (or the number of expected revenue "ranks") to be equal to the number of states achievable when $Z = I$. Note that this could mean that two or more states could correspond to the same expected revenue value. This is without loss of generality because, by considering the number of states this way, we can have a threshold $p = \bar{p}(Z)$ for every $Z \subseteq I$ such that $\Pi(Z, \bar{p}(Z)) = \Pi(Z, \bar{p}(Z) + 1) = \Pi(Z, \bar{p}(Z) + 2) = \cdots = \Pi(Z, L)$ and for $Z, Z' \subseteq I$ where $|Z| \neq |Z'|$, we could have that $\bar{p}(Z) \neq \bar{p}(Z')$, where $\Pi(Z, \bar{p}(Z))$ is the auctioneer's expected revenue when she puts up item set Z for sale.

The size of the state space for a given item set $Z \subseteq I$ is at least exponential in the cardinality of Z. In fact, even under the assumption that $n_b = 1$, it is the number of non-negative integral solutions to the following equation in variable $u_i,\ i \in I$

$$\sum_{i \in I} u_i = |B|, \tag{4.1}$$

which is $\binom{|B| + |I| - 1}{|I| - 1}$. Here, the value of $u_i,\ i \in I$ can be interpreted as the number of bidders bidding for item $i,\ i \in I$. The number of non-negative integral solutions to the above equation is equal to the number of possible ways in which the bidders participate in the auction for the set of offered items I (i.e., the possible participation patterns possible when set I is put up for sale to $|B|$ bidders) under the assumption that each bidder only bids for one item. Thus, when $n_b \geqslant 1,\ \forall b \in B$, the size of the state space is very large. Given such a large state space, the expected revenue from the sale of a set Z of items, with the expectation being taken over all possible states and over the second-highest bid for each item given a state, is computationally very difficult for the auctioneer to calculate.

4.3.2 The Minimax Regret Criterion

Let $\Pi(Z, p)$ be the auctioneer's expected revenue when she puts up set $Z \subseteq I$ of items for sale, and the participation pattern she will observe on those items is $p \in \mathcal{P} = \{H, 2, 3, 4, \cdots, L\}$. The regret $Reg(Z, p)$ of offering a set Z when the state is p is calculated as

$$Reg(Z, p) = \max_{Y \subseteq I} \Pi(Y, p) - \Pi(Z, p) \tag{4.2}$$

The auctioneer considers all states $p \in \{H, 2, 3, 4, \cdots, L\}$ and evaluates the maximum regret for each choice Z

$$\max_{p} Reg(Z, p) \quad \forall Z \subseteq I \tag{4.3}$$

The auctioneer selects a strategy Z^* where

$$Z^* = \arg\min_{Z \subseteq I} \max_{p} Reg(Z, p) \tag{4.4}$$

We refer to the problem of computing Z^* in the above manner as the *Optimal Offer Set Selection* problem (OOSS). Our use of the minimax regret criterion stems from how the term "regret" can be interpreted to mean "cost of uncertainty" in the context of our problem, and also from the drawbacks of other approaches to take into account uncertainty: The expected-revenue-maximization approach is not easy in practice as it involves the computation of the mixed strategy Nash equilibria of the participation decisions of the bidders, and is therefore not easy to use in practice. Because of this, we have chosen a minimax regret criterion as it models the "cost of uncertainty" appropriately in our context.

4.4 Main Results

In this section, we present several results necessary for the efficient computation of Z^* in §4.3.2. Each of these results offer useful insights into the structure of the OOSS.

4.4.1 The fundamental optimization problem

The computation of $\Pi(Z, p = H)$, where $Z \subseteq I$, is fundamental to our results. $\Pi(Z, p = H)$ can be computed by solving the following optimization problem. We call the following problem COMP(Z).

$$\textbf{COMP(Z):} \qquad \max_{\mathbf{y}} \qquad G(\mathbf{y}) = \sum_{i \in Z} ER_i\Big(\sum_{b \in B} y_{ib}\Big) \tag{4.5}$$

$$\text{subject to} \qquad \sum_{i \in Z} y_{ib} = N_{bU} \quad \forall \ b \in B \tag{4.6}$$

$$y_{ib} \in \{0, 1\} \quad \forall \ i \in Z, \ \forall \ b \in B \tag{4.7}$$

where

$$ER_i(n) = E[\max\{X_{i(2:n)}, r_i\}], \tag{4.8}$$

$\mathbf{y} = \langle y_{ib}, \ i \in I, \ b \in B\rangle$, and $X_{i(2:n)}$ is the second-highest random variable in a sample of n random variables sampled from a distribution with CDF F_i and $r_i = l_i$ is the reserve price for item i. We do not go into computing reserve prices for items. The quantity $E[\max\{X_{i(2:n)}, r_i\}]$ is the expected value of the maximum of $X_{i(2:n)}$ and r_i. The term $\sum_{b \in B} y_{ib}$ in the argument of $ER_i(\cdot)$ in (4.5) is the total number of bidders participating in the auction for item i. The objective function (4.5) is the total expected revenue the auctioneer makes by conducting a simultaneous second-price sealed-bid auction of items in the set Z. Constraint (4.6) constrains the number of items bidder b can inspect and subsequently bid on. Since we consider the state $p = H$, we set the RHS of constraint (4.6) to its highest value possible, i.e., $N_{bU}, \ b \in B$. This is because setting the RHS of constraint (4.6) to $N_{bU}, \ b \in B$ maximizes bidder participation since their inspection budgets are set their respective maximum levels. Constraint (4.7) restricts $y_{ib}, \ i \in Z, \ b \in B$ to 0 or 1, where $y_{ib} = 1$ indicates that bidder b bids for item i, and $y_{ib} = 0$ indicates that bidder b does not bid for item i. We assume throughout this chapter that the function $ER_i(\cdot)$ can be computed in constant time.

We now present a polynomial-time greedy heuristic GH to solve COMP(Z).

Algorithm 2 GH

1. Initialize $y_{ib} = 0, \ \forall \ i \in Z, \ b \in B$

2. **while** $\sum_{i \in Z} y_{ib} \neq N_{bU}, \ \forall \ b \in B$ **do**

    3. **for** $i \in I, \ b \in B$ **do**

        4. **if** $y_{ib} = 0$ **then**

            5. **if** $(i, b) : (i, b) = \arg\max_{(j,c)} G(\mathbf{y}|y_{jc} = 1) - G(\mathbf{y}|y_{jc} = 0)$ **then**

                6. Set $y_{ib} = 1$

            end

        end

    end

end

7. Show $y_{ib}, \ i \in I, \ b \in B$

In Line 5, the term $G(\mathbf{y}|y_{jc} = h)$, where $h \in \{0, 1\}$, refers to the value of the objective function (4.5) when all components of $\mathbf{y}$, except component y_{jc}, are held at their current values, and y_{jc} is set to value $h, \ h \in \{0, 1\}$.

Algorithm GH starts by setting $y_{ib} = 0$ for all $i \in Z, b \in B$. Then, it selects an arbitrary bidder b and sets $y_{ib} = 1$ if doing so results in the maximum increase in the objective function value (4.5) compared to when $y_{ib} = 0$. Next, it selects a bidder b', where b' may or may not be b, and sets $y_{i'b'} = 1$, where i' may or may not be equal to i when $b' \neq b$, if doing so results in the maximum increase in the objective function value (4.5) compared to when $y_{i'b'} = 0$ while fixing $y_{ib} = 1$. It repeats until $\sum_{i \in Z} y_{ib} = N_{bU}$ for all $b \in B$. We now have the following result regarding the computation time of GH.

Proposition 4.1 *GH is an exact algorithm, and COMP(Z) can be solved in* $O(|B||I|^2)$ *steps.*

Proof of Proposition 4.1

First, we note that the objective function (4.5) is separable in i. Secondly, the quantity $ER_i(x+1) - ER_i(x) \leqslant ER_i(x) - ER_i(x-1), \forall x \in$ Integers , i.e., the quantity $ER_i(x+1) - ER_i(x)$ decreases as x increases. This follows from the fact that the random variables $X_{ib},\ b \in B$ are IID for each given $i \in B$ and that $F_i,\ i \in I$ is regular. A proof of this is provided in Overby and Kannan (2015). Thirdly, we note that the quantity $\sum_{b \in B} y_{ib}$ (in the argument of function $ER_i(\cdot)$ for each $i \in Z$ in objective function (4.5)) is the number of bidders who participate in the auction for item i.

At every step of GH, a decision $d(i', i'', b')$ needs to be made on whether to set $y_{i'b'} = 1$ or if $y_{i''b'} = 1$, i.e., a choice needs to be made as to whether bidder b' should be assigned to item i' or item i''. Let the number of assigned bidders to item i' when decision $d(i', i'', b')$ needs to be made be $x^*_{i'}$ and let number of assigned bidders to item i'' when decision $d(i', i'', b')$ needs to be made be $x^*_{i''}$.

The marginal gain in objective function (4.5) when $y_{i'b'} = 1$ is $ER_{i'}(x^*_{i'} + 1) - ER_{i'}(x^*_{i'})$, and The marginal gain in objective function (4.5) when $y_{i''b'} = 1$ is $ER_{i''}(x^*_{i''} + 1) - ER_{i''}(x^*_{i'})$. Now, GH would choose $y_{i'b'} = 1$ if $ER_{i'}(x^*_{i'} + 1) - ER_{i'}(x^*_{i'}) \geqslant ER_{i''}(x^*_{i''} + 1) - ER_{i''}(x^*_{i'})$. As a result, at the end of every step of GH, there is an improvement in the objective function (4.5), and this the maximum improvement from the value of objective function (4.5) in the previous step. Also, there is a finite number of such improvement steps. Since, for each item i, the bidders' values of are IID, GH results in convergence to the optimal solution to problem COMP(Z).

We noted earlier that GH is a polynomial time heuristic, and we showed here that GH converges to the optimal solution to problem COMP(Z). Thus, there exists a polynomial time algorithm to solve problem COMP(Z).

To show that the number is steps is $O(|B||I|^2)$, we note that GH makes a total of $|I||B|$ assignments. Also, it takes $O(|I|)$ steps to compute the optimal assignment y_{ib} for a given b, since it requires searching over at the most $|I|$ items to identify the assignment that contributes the highest marginal increase in objective function. Therefore, GH takes $O(|B||I|^2)$ steps to terminate. □

Thus, the auctioneer's maximum expected revenue from offering a set $Z \subseteq I$ for sale can be computed in polynomial time. We now look at some important results needed for solving OOSS.

4.4.2 Important Observations and Results

We present the following result from Lawler (1972) for a binary integer optimization problem with m binary variables.

Proposition 4.2 *(Originally from (Lawler 1972); restated here) If the number of steps required to compute a solution to a single binary optimization problem with m variables is $c(m)$, then the number of computational steps required to compute the K best solutions is $O(Kmc(m))$.*

Computing $\Pi(Z, p)$ for states other than $p = H$.

We make use of the solution to $\Pi(Z, p = H)$ (computed using GH) to compute $\Pi(Z, p)$ where $p \neq H$. In particular, we make use of the approach discussed in Lawler (1972). However, computing $\Pi(Z, p)$ when $p \in \{2, 3, \cdots, L\}$ can take a very long time. This is because, as we noted in §4.3.1, the number of states is at least exponential in size.

Computing $\Pi(Z, p = L)$.

This would involve solving problem (4.5), (4.6), (4.7) but with a minimization objective function. This, too, can be solved in polynomial time by identifying an assignment resulting in the minimum increase in Line 5 instead of the maximum increase. This solution results in the worst-case expected revenue for the auctioneer if item set Z is put up for sale. We now present some structural results.

Lemma 4.1 *Consider two sets of items $Y_1 \subseteq I$ and $Y_2 \subseteq I$ such that $|Y_1| = |Y_2|$. Consider a natural number n, where $n = |B|$. Let $i_1 = \arg\max_{i \in Y_1} ER_i(n)$ and let $i_2 = \arg\min_{i \in Y_2} ER_i(n)$. For any p, let $\Pi(Y, p)$ be the auctioneer's expected profit when set Y of items is sold to set B of bidders. If Y_1 and Y_2 are such that $ER_{i_1}(n) \leqslant ER_{i_2}(n)$, then it is always true that $\Pi(Y_1, p) \leqslant \Pi(Y_2, p)$ and $\max_{Y \subseteq Y_1} \Pi(Y, p) \leqslant \max_{Y \subseteq Y_2} \Pi(Y, p)$.*

Lemma 4.1 essentially states that for a given natural number n if one considers two sets (say Y_1 and Y_2) of equal cardinality and a given state p, the item set (say Y_2) containing the items, which, if auctioned off singly to n bidders (each of whom can only acquire one item), would yield a higher expected revenue to the auctioneer compared to the item set (say Y_1) containing the items, which, if auctioned off singly to n bidders, would yield a lower expected revenue to the auctioneer. This follows from the fact that the items in the set Y_2 would receive bids the second-highest of which (or the reserve price) is higher in expectation than the bids the items in set Y_1 receive. Therefore, for a given state p, selling items from set Y_2 on auction would give the auctioneer a higher expected revenue than selling items from set Y_1.

Intuitively, Lemma 4.1 refers to the fact that putting up a set of items (of a given cardinality) that are "more valuable" for sale to a given number of bidders would provide the auctioneer with higher revenue in expectation compared the expected revenue from putting up a set of items (with the same cardinality) that are "less valuable" to the same set of bidders.

Lemma 4.2 *Let* $Y^{**} = \arg\max_{Y \subseteq I} \Pi(Y, p = H)$. *Then,* $Y^*(p) \equiv \arg\max_{Y \subseteq I} \Pi(Y, p)$ *for all* $p = \{H, 2, 3, \cdots, L\}$ *is such that* $Y^*(p) \subseteq Y^{**}$.

Note that the set Y^{**} is such that for every $j \in I - Y^{**}$, there exists $i \in Y^{**}$ such that $ER_i(n) \geqslant ER_j(n)$ for $n = |B|$. This follows from the way the set Y^{**} is chosen using algorithm GH. As a result of Lemma 4.2, for any state p, it is the subsets Y of the set Y^{**} that will maximize $\Pi(Y, p)$ and not the subsets of any other set $I - \{Y^{**}\}$. Thus, this result provides a way to reduce the search space for solving the OOSS. Note that $Y^{**} \subseteq I$. Thus, a reduction of search space from I to a strict subset of I may not be possible for all problem instances.

Lemma 4.3 *Consider two sets of items* $Y_1 \subseteq I$ *and* $Y_2 \subseteq I$ *such that* $|Y_1| = |Y_2|$. *Consider a natural number* $n = |B|$. *Let* $i_1 = \arg\max_{i \in Y_1} ER_{i_1}(n)$ *and let* $i_2 = \arg\min_{i \in Y_2} ER_{i_2}(n)$. *If* Y_1 *and* Y_2 *are such that* $ER_{i_1}(n) \leqslant ER_{i_2}(n)$. *For any* p, *let* $Reg(Y, p)$ *be the auctioneer's regret associated with selling item set* Y *to bidders in set* $|B|$. *Then, for any* p, *it is always true that* $Reg(Y_1, p) \geqslant Reg(Y_2, p)$.

Lemma 4.3 follows directly from Lemma 4.1. Intuitively, the expected revenue from putting up a set of items with higher bidder valuations (in expectation) is higher than the expected revenue from putting up a set of items with lower bidder valuations (in expectation). Therefore, set Y_2 yields a lower regret to the auctioneer. We now define the set $\bar{I}$ constructed as follows:

1. Initialize $\bar{I} = \{\}$, Initialize $J = Y^{**}$
2. $\bar{I} \leftarrow \bar{I} \cup J$.
3. Let $i^* = \arg\min_{i \in J} ER_i(n)$ where $n = |B|$.
4. $J \leftarrow J - \{i^*\}$
5. Go to step 2 until J is empty

It is important to note Step 3 and Step 4 above. We remove the item i^* and create a new set J without item i^*. Repeating this process creates a set of item sets $\bar{I}$ with the following property: No subset of I that does not belong to $\bar{I}$ will solve OOSS, as we see from Lemma 4.3. This is because for every subset of $Y^{**} \subseteq I$ of a given cardinality, there exists an element of set $\bar{I}$, which, if sold to bidders in B would yield a higher or equal expected revenue for all states $p \in \mathcal{P}$. This follows from Lemma 4.1. Thus, the set $\bar{I}$ only contains those sets of items that solve OOSS. Therefore, only these item sets need be examined as possible solutions to OOSS.

The computational complexity of computing the regret associated with an offer set Z. From §4.3.2, the regret $R(Z, p)$ from offering a set Z when the state is p is calculated according to equation (4.2). The computation of $\Pi(Z, p)$ in equation (4.2) will first involve the computation of $\Pi(Z, p = H)$. This can be done in polynomial time using algorithm GH.

Examining the problem of computing $\Pi(Z, p)$ for any $p \in \mathcal{P}, p \neq H$ in terms of Proposition 4.2, we have that $m = |Z||B|$ and from Proposition 4.1, we have that $c(m) = |B||I|^2$. If $K = |\mathcal{P}|$ is the number of states, the time taken for computing

$\Pi(Z, p)$ for all p is $O(K|Z||B|^2|I|^2)$ and is exponentially large since $K = |\mathcal{P}|$ is exponentially large. This makes the computation of the minimax regret computationally difficult despite only having to search for the optimal strategy Z^* in the set $\bar{I}$.

Since the computational complexity of OOSS depends crucially on the cardinality of the set $\mathcal{P}$, restricting the elements of of set $\mathcal{P}$ to take on values within a polynomially bounded range of H and L can ensure polynomial time computations. Thus, if $|\mathcal{P}|$ were polynomially bounded, computing OOSS is possible in polynomial time.

4.5 Solving the OOSS if the state space is polynomially bounded

As noted at the end of §4.4.2, restricting the state space $\mathcal{P}$ such that $\mathcal{P} = \{H, L\}$ will speed up computation time at the cost of being able to find an exact solution. If we only consider that $p \in \{H, L\}$ and ignore all the other states, the following algorithm solves OOSS in polynomial time.

Algorithm 3 Solving OOSS under restricted state-space $\mathcal{P}$.

1. Initialize $J = I$
2. **while** $J \neq \{\}$ **do**
   3. Compute $Reg(J, p)$ for $p = \{H, L\}$
   4. Compute $\max_{p \in \{H,L\}} Reg(J, p)$
   5. Set $j = \arg\min_{j \in J} ER_j(n)$ where $n = |B|$
   6. $J \leftarrow J - \{j\}$

end

7. $Z^* = J^* = \arg\min_{J \in \bar{I}} \max_{p \in \{H,L\}} Reg(J, p)$

Notice that J in step 6 is such that $J \in \bar{I}$. Note that the cardinality of $\bar{I}$ is $|I|$. For all $J \in \bar{I}$, note that the values of $\max_{p \in \{H,L\}} Reg(J, p)$ are computed in step 4 for all $J \in \bar{I}$. Computing $\max_{p \in \{H,L\}} Reg(J, p)$ for a given $J \in \bar{I}$ is possible in polynomial time. The final step 7 can be performed in polynomial time.

Note that we only need to search over all $J \in \bar{I}$ and not all subsets of I because of Lemmas 4.1, 4.2, and 4.3. This is exactly what we do in Algorithm 3. For each $J \in \bar{I}$, we perform a polynomial number of operations in step 3 and step 4. The final step is a polynomial time step. In general, if $|\mathcal{P}|$ is polynomially bounded, Algorithm 3 terminates in polynomial time.

We now solve the OOSS with Algorithm 3 for the problem instance discussed in §4.1 by only considering two states $\{H, L\}$. To restate the problem in terms of our notation, we have $I = 1, 2, 3, B = 1, 2, 3, l_1 = 1, l_2 = 2, l_3 = 3, u_1 = 2, u_2 = 3, u_3 = 3$. Also, $X_{ib},\ b \in B$ are uniformly distributed with support $[l_i, u_i],\ i \in I$.

When $p = H$, then

$$J(p = H) = \arg\max_{Y \subseteq I} \Pi(Y, p = H) = \{1, 2, 3\} \tag{4.9}$$

$$\Pi(J(p = H), p = H) = 1 + \frac{2}{4} + 2 + \frac{2}{4} + 3 + \frac{2}{4} = 7.5 \tag{4.10}$$

When $p = L$, then

$$J(p = L) = \arg\max_{Y \subseteq I} \Pi(Y, p = H) = \{3\} \tag{4.11}$$

$$\Pi(J(p = L), p = L) = 3 + \frac{8}{10} = 3.8 \tag{4.12}$$

For a given $Z \subseteq I$, we have

$$Reg(Z, p = H) = 7.5 - \Pi(Z, p = H)$$

$$Reg(Z, p = L) = 3.8 - \Pi(Z, p = L)$$

When $Z = \{1, 2, 3\}$

$$Reg(Z, p = H) = 7.5 - \Pi(Z, p = H) = 7.5 - 7.5 = 0$$

$$Reg(Z, p = L) = 3.8 - \Pi(Z, p = L) = 3.8 - 1.8 = 2$$

The maximum regret of $Z = \{1, 2, 3\}$ is 2 units.

When $Z = \{2, 3\}$

$$Reg(Z, p = H) = 7.5 - \Pi(Z, p = H) = 7.5 - (2 + \frac{3}{5}) - (3 + \frac{4}{6}) = 1.233$$

$$Reg(Z, p = L) = 3.8 - \Pi(Z, p = L) = 3.8 - (2 + \frac{8}{10}) = 1,$$

The maximum regret of $Z = \{2, 3\}$ is 1.233 units.

When $Z = \{3\}$

$$Reg(Z, p = H) = 7.5 - \Pi(Z, p = H) = 7.5 - 3.8 = 3.7$$

$$Reg(Z, p = L) = 3.8 - \Pi(Z, p = L) = 3.8 - 3.8 = 0$$

The maximum regret of $Z = \{3\}$ is 3.7 unit.

When $Z = \{2, 3\}$, we obtain the lowest maximum regret of 1.233 units. Thus, the solution to the OOSS problem instance in the illustration in §4.1 is $Z^* = \{2, 3\}$.

Thus, if the auctioneer (decision maker) only considered a state space with a polynomially-bounded size, the problem of computing the optimal offer set Z^* is solvable in polynomial time, as demonstrated in this example.

4.6 Conclusions and Future Research

We examined the problem of optimizing offer sets for sealed-bid multi-item simultaneous auction settings. The importance of this problem arises from the fact that the participation decisions of the bidders, which we referred to as *participation patterns*, is unknown to the auctioneer for any subset of the item set she puts up for sale.

The results of described here apply to several business settings that involve the simultaneous sealed-bid auctions of multiple items. These include auctions for resource tracts and also online auctions.

In this regard, we used a minimax regret criterion to account for the uncertainty and solve the problem of identifying the optimal offer set. A key modeling construct we made use was that, given a subset of items put up for sale, each participation pattern corresponded to a value of expected revenue. As a result, we were able to rank-order the states of the system using the values of the expected revenue, where each value of expected revenue corresponded to a state.

Our results show that when the bidders' item valuations are additive and when the bidders' valuations for an item are independently and identically distributed, the problem of identifying the max-regret minimizing subset of items can be solved in polynomial time if the state space considered is polynomially bounded.

To the best of our knowledge, a problem of this nature has not been examined in literature before. Our work is the first to have examined a problem of this nature. Given the value of the transactions involved in the business settings where our results apply, we believe brought to light a major gap in literature that is of relevance to practice. This leaves room for several streams of literature in future. Some potential extensions are as follows:

1. We examined a single period problem of optimizing offer sets. However, in many settings in practice, auctions are conducted periodically such as the US oil and natural gas auctions (Hendricks et. al. 2014). In such settings, the problem of optimizing the offer set for the current period will depend on the expected revenues from the future period. This has some overlaps with the multi-period asset-selling problem. Our belief is that this could lead to an explosion of the state space size required for each period.

2. We examined a setting where bidder valuations for an item are additive. One could consider a setting where items are complements or substitutes. We point out, though, that in such settings, the notion of a second-price simultaneous sealed-bid auction is ill-defined since this auction format does not consider overlaps in bundles. Thus, one may consider examining the problem under other auction settings with different rules. The common denominator with our setting is the uncertainty in participation patterns that the auctioneer

faces when she puts up an item for sale.

3. A third possibility is that of bundling the items before putting them up for sale under a simultaneous second-price sealed bid auction. In such cases, computing the optimal bundling would be an interesting problem to examine.

Notes

[9] To see how this comes about, note that if $P_1, P_2, \cdots, P_n$ are n IID random variables uniformly distributed in $[a, b]$, then the expected value of the second-highest random variable is $a + \frac{n-1}{n+1}b$

[10] There are two levels to the 'expectation' operator: The first one is the auctioneer's expected revenue for a given participation pattern, and the second one is the expectation computed around the uncertain participation patterns.

Chapter 5

Auction Mechanisms for Social-Welfare-Maximizing Allocations with Pairwise-Additive Negative Value Externalities

5.1 Introduction

Online advertising service providers (or platforms) typically use auctions to sell advertising rights to advertisers. Well-known examples are Google's sponsored search advertising and Facebook's sponsored advertisements. Platforms auction display positions (or slots) on their webpages to advertisers who, in turn, place their advertisements in these slots. The value of an advertisement slot on a webpage to an advertiser is influenced by the allocation of the other advertisement slots on the same webpage to other advertisers. Such influences are known as "externalities" (Constantin et. al. 2011, Bhargava et. al. 2019, Sayedi et. al. 2018). The impact of these externalities is often negative, and has prompted a rise in the demand for exclusivity in advertising (Sayedi et. al. 2018, Bhargava et. al. 2019). Advertisers may be willing to pay more for slot allocations where the impact of such negative externalities is minimal (Bhargava et. al. 2019). The authors present a simple argument for this phenomenon: advertisers value exclusive allocations more than shared allocations, and are willing to pay more for such exclusive allocations. Thus, when a set of indivisible items[11] (advertisement slots) are allocated to bidders[12] (advertisers), the value of each item to a bidder depends on how the other items are allocated to the other bidders.

Typically, negative allocative externalities are categorized as follows: (*i*) quantity externalities, where the magnitude of the externalities is observable to the auctioneer, and (*ii*) value externalities, which are private to a bidder and not observable to the auctioneer or to the other bidders (see Constantin et. al. 2011, and other references therein for more details). Our work focuses on the latter type of allocative externalities, i.e., negative value externalities.

Here, I consider the auctioneer's[13] problem of (*i*) allocating a finite number of heterogeneous and indivisible items to a set of bidders, where (*ii*) an item can only be assigned to *one* bidder, when (*iii*) the bidders' valuation functions account for negative value externalities, and (*iv*) the allocations need to maximize social welfare. In particular, I on focus auction designs (direct and indirect mechanisms) that result in such allocations.

In this regard, I define a notion of an equilibrium outcome (defined by an allocation and a set of item-prices), applicable to market design considering allocative externalities. This notion of equilibrium is similar to a Walrasian equilibrium (Mas-Collel et. al. 1995), but can be extended to settings where the bidders' functions consider allocative externalities. If item prices in an iterative auction are modified to lead to an equilibrium allocation, no bidder is better off with an allocation different from the equilibrium allocation. I show that, in general, for a given set of bidder valuation functions, an equilibrium with prices set at the item level does not exist. In this regard, I present results on the conditions on the bidders' valuation functions under which an equilibrium defined by item-level prices exists.

As our main contribution, I study a class of valuation functions, that I call *Pair-*

wise Additive Negative Value Externalities (PANE), wherein a bidder's valuation of an item has two parts: (*i*) a portion independent of all other allocations, and (*ii*) the negative externality, by which the item's value to the bidder decreases, when rival bidders are assigned other items. To each bidder, the magnitude of the total negative externality associated with an item is the sum of the externalities imposed by the assignments of the other items to the other bidders.

The auctioneer's goal is find an allocation that maximizes the social welfare under such a valuation function. To this end, an auctioneer may consider using a direct mechanism or an indirect mechanism to achieve the goal. A Vickrey-Clarke-Groves (VCG) mechanism is a direct mechanism that can be used to achieve a social-welfare-maximizing allocation. By design, the bidders are incentivized to report truthful bids in one shot, following which the auctioneer decides on an allocation that maximizes social welfare. An indirect mechanism (such as an iterative auction) is one where the auctioneer seeks information, other than bidder preferences for items, that can be used to achieve a social-welfare-maximizing allocation. This information is obtained by noting bidder behavior as a response to item prices. Typically, after the auctioneer sets and announces item prices, the bidders report their interests in the items (or item bundles) in response to these prices. The auctioneer notes the responses from the bidders and modifies item (or bundle) prices accordingly to achieve an item allocation that maximizes social welfare.

Given the bidders' truthful valuation functions, the practical implementability of the VCG mechanism (direct mechanism) to achieve social-welfare-maximizing allocation, depends on whether the auctioneer's ability to compute the social-welfare-maximizing allocation quickly. Thus, it is important to know if the social-welfare-maximizing allocation under PANE can be computed efficiently. The practical implementability of an iterative auction (indirect mechanism) to achieve a social-welfare-maximizing allocation depends on whether there exists an equilibrium with simple pricing schemes (such as item-level pricing schemes) under the PANE valuation function. That is, since indirect mechanisms note bidder behavior as a response to prices, it is important to know if the pricing schemes to needed to achieve a social-welfare-maximizing allocation under PANE are simple for practical implementation (such as item-level pricing schemes). This is because iterative auctions with item-level prices need not necessarily lead to a social-welfare-maximizing allocation under general valuation functions. In this chapter, we answer these key questions about the PANE valuation function.

I formulate the problem of computing a social-welfare-maximizing allocation under PANE as a quadratic binary integer program (QIP), which is a nonconvex optimization problem. General instances of such formulations are known to be NP-Hard (Zheng et. al. 2012, Pardalos et. al. 1991). However, my formulation demonstrates specific structural properties that allow us to solve the maximization problem efficiently and develop structural insights into auction designs that ensure social-welfare-maximizing allocations while maintaining computational tractability. I show that the continuous relaxation of the social-welfare maximization problem yields integral solutions. Additionally, we show that strong duality holds true for the relaxed problem that helps solve the problem in polynomial time using an equivalent semidefinite formulation. Thus, the optimal solution to the social welfare maximization problem can be obtained in polynomial time by solving the relaxed social welfare maximization problem. Consequently, the computations on allocations and payments required for the VCG mechanism (the direct mechanism) can be performed efficiently. This result is of useful for online platforms that require quick computation of optimal allocations and payments.

My structural results on integrality and strong duality imply the existence of *anonymous* and *simple* equilibrium item prices. In other words, I show that the equilibrium prices for this class of valuation functions do not depend on the identity of the bidder (anonymous). Also, the prices are only associated with individual items and not bundles of items (simple)[14]. This allows us to show that under PANE, there exists an equilibrium allocation with prices that are set at the item-level, i.e., every (indivisible) item is allocated to one bidder and each bidder's surplus (the difference between a bidder's value for a subset of items and price of the subset of items) is maximized.

Building on the strong duality result and on the fact that an equilibrium with item-level prices exists, I show that a subgradient algorithm (Fisher 2004) for solving QIP converges to the social-welfare-maximizing allocation. The subgradient algorithm may be interpreted as an auction where prices of items are decided based on the interest for the items at a given set of prices. I refer to this format as a subgradient auction or a guided-price auction. This is interesting because, unlike in the case of a simultaneous sale of multiple items with no externalities, it is not immediately obvious how an auction format that guides prices can bring about a social-welfare-maximizing allocation. Essentially, our auction format converges to the social-welfare-maximizing allocation by guiding item price movements based on the interest for the items. The advantages of such an auction format are that (*i*) the winners reveal less information, (*ii*) it creates a perception of fairness because the low information revelation requirements mean that the auctioneer cannot manipulate the bidders, and (*iii*) the burden of computation and the communication of valuations is low, since the bidders only need to respond to the current prices of the items (Cramton 1998, de Vries et. al. 2007). This auction format thus serves as an indirect mechanism that terminates at an allocation that maximizes social welfare. It is noteworthy that this auction format makes use of simple and anonymous prices.

I summarize my contribution as follows: (*i*) I introduce a notion of equilibrium for settings with allocative externalities, (*ii*) I present a general condition on the bidders' valuation functions for which simple and anonymous prices are sufficient to support a social-welfare-maximizing allocation, (*iii*) I identify a class of linearly constrained binary quadratic programs whose continuous relaxations yield binary solutions, (*iv*) I identify a class of linearly constrained continuous nonconvex quadratic optimization problems where strong duality holds, enabling polynomial time solvability of such a class of optimization problems, and (*v*) I identify a class of valuation functions with negative externalities where the social-welfare-maximizing allocation can be achieved through a known auction format using simple and anonymous pricing structures.

The rest of the chapter is organized as follows: In §5.2, I discuss literature on auction design with allocative externalities in the single and multi-item settings. I also discuss literature on the connection between duality theory of mathematical programming and multi-item auction design. In §5.3, I formally present the model assumptions, describe a notion of equilibrium for settings with allocative externalities, and formally describe the Pairwise-Additive Negative Value Externalities (PANE) valuation function. In §5.4, I present our key technical results on social welfare maximization under PANE. In §5.5, I present a subgradient algorithm to solve the social welfare maximization problem under PANE and show how the subgradient algorithm can be represented as a multi-round guided-price auction. In §5.6, I present some numerical demonstrations of strong duality of the relaxation of the social-welfare maximization problem under PANE, and also present illustrations on how the guided-price auction works under PANE. Finally, in §5.7, I summarize the chapter and present directions for future work.

5.2 Literature Review

I begin by highlighting the literature related to single-item auctions and externalities, followed by literature related to multi-item auction design under externalities. Finally, I also highlight literature related to the association between dual prices and auction theory.

Externalities in single-item auctions. One of the earliest works on auctions with negative allocative externalities is that of Jehiel et. al. (1996). In this paper, the authors illustrate how the presence of allocative externalities can affect the auctioneer's surplus in a general business context. They consider a single-item auction for their analysis. Belloni et. al. (2017) study the problem for designing the optimal mechanism for a single-item setting with private externalities. They discuss the dependence of the auctioneer's revenue on the magnitude of the externalities. An insight from these papers is that bidders may find it rational to pay even if they don't receive the item since the outcome of the item going into the hands of a rival is a suboptimal outcome.

Externalities in multi-item auctions. There are several papers associated with advertisement slot auctions or online auctions in general. Krysta et. al. (2010) present a framework for the analysis of multi-item auctions with allocative externalities and show some results on the computational complexity of the allocation problem in general settings. Cheung et. al. (2015) discuss combinatorial auctions with externalities and present approximation algorithms for the problem of maximizing social welfare. Deng et. al. (2011) show that in a multi-item setting with allocative externalities, there may exist allocative equilibria in which no bidder gets any item with the auctioneer making a positive surplus. Ghosh and Mahdian (2008) examine the problem of winner determination in a general valuation setting, and show how the computational complexity of the problem makes approximation hard. Ghosh and Sayedi (2010) discuss the issues of externalities in online advertising, and propose a bidding language that allows bidders to express their valuations for the items along with the externalities. The authors the analyse allocative efficiency of a GSP[15]-like mechanism and a VCG-like mechanism they propose for use with this bidding language. Gomes et. al. (2009) conduct an empirical analysis of the magnitude of the externalities induced by competing advertisement links in advertisement slot auctions. Their results show that the impact of externalities is statistically significant. Zhang et. al. (2018) study the problem of computing a social-welfare-maximizing allocation of items in a setting with identity-based negative externalities where there is a single item with unlimited supply. In their model of externalities, a bidder's private valuation from winning decreases as the number of rivals increases.

Duality and auction theory. Bikhchandani et. al. (2002) present a detailed analysis of the connections between linear programming, duality theory, and auction theory in the context of multi-item auctions. In particular, they interpret the duals of the social-welfare-maximizing problem as prices of the items on sale. They show that a set of prices resulting in a social-welfare-maximizing allocation always exist, and show how the complexity of the bidders' valuation functions for the items results in non-linear (complex) prices for items. The doctoral dissertation of Parkes (2001) presents a detailed analysis of the connection between duality theory and the social welfare maximization problem. Candogan et. al. (2015) show how a social welfare maximization problem, in settings where the magnitude of item valuations and complementarities (or substitutabilities) can be represented in the form of a tree, can be solved as a linear program. The authors show the connection between the duals of this linear program and the payments associated with social-welfare-maximizing allocation.

5.3 Model and Preliminaries

Let B represent the set of bidders and L the set of heterogeneous, indivisible items. The bidders in B seek to acquire some or all the items in L. Let BL be the Cartesian product of B and L. The allocation decision vector, $\mathbf{x} \equiv \langle x_i^a,\ (i,a) \in BL \rangle$, is such that each component $x_i^a \in \{0,1\}$ for all $(i,a) \in BL$ is defined as follows.

$$x_i^a = \begin{cases} 0: & \text{if item } a \in L \text{ is not assigned to bidder } i, \\ 1: & \text{if item } a \in L \text{ is assigned to bidder } i. \end{cases} \tag{5.1}$$

$\mathbf{x}$ is a *feasible allocation* of items if $\sum_{i \in B} x_i^a = 1$ for all $a \in L$, i.e., each item $a \in L$ is assigned to exactly one bidder $i \in B$.

Let $v_i(\mathbf{x})$ represent the valuation function corresponding to a feasible allocation $\mathbf{x}$ and $S_i(\mathbf{x})$ is the set of items bidder i receives under allocation $\mathbf{x}$. We assume that $v_i(\mathbf{x})$ is privately known to bidder i for all $i \in B$ and $v_i(\cdot) : [0,1]^{|BL|} \to \Re$ for all $i \in B$ where $\Re$ is the set of real numbers. I also assume that $v_i(\cdot)$ is differentiable everywhere for all $i \in B$. The social welfare of the bidders for any allocation $\mathbf{x}$ is defined as $\sum_{i \in B} v_i(\mathbf{x})$.

5.3.1 Social Welfare Maximization Under Negative Value Externalities

Negative value externalities are said to exist in an allocation $\mathbf{x}$ when the following conditions are satisfied (Constantin et. al. 2011): (i) bidder i's private value for an item a depends also on the allocation of the other items $b \in L,\ b \neq a$ to the other bidders $j \in B,\ j \neq i$, and (ii) information on the magnitude of this dependence is private. This is in contrast to quantity externalities, which are observable to the auctioneer (Constantin et. al. 2011). The general social welfare maximization problem (GSWMP) under negative value externalities is shown in (5.2) – (5.4). The GSWMP aims to find a feasible allocation that maximizes the sum of the bidders' valuations from the allocation.

$$\textbf{GSWMP:} \qquad \max \qquad H(\mathbf{x}) = \sum_{i \in B} \Bigg[v_i(\mathbf{x}) \Bigg], \tag{5.2}$$

$$\text{subject to} \qquad \sum_{i \in B} x_i^a = 1 \quad \forall\ a \in L, \tag{5.3}$$

$$x_i^a \in \{0, 1\} \quad \forall\ (i, a) \in BL. \tag{5.4}$$

The objective (5.2) maximizes the social welfare of the bidders. Constraint (5.3) ensures that an item is only assigned to one bidder. Constraint (5.4) ensures that $\mathbf{x} = \langle x_i^a,\ (i, a) \in BL \rangle$ has binary-valued components at optimality because the items being auctioned are assumed to be indivisible. It is noteworthy that GSWMP always has an optimal solution. This is because the feasible region (5.3) – (5.4) is a non-empty, compact, polyhedral set.

As mentioned in §5.1, one of my goals is to design an indirect iterative auction mechanism that yields a social-welfare-maximizing allocation upon termination. In designing such an auction mechanism, it is desirable that (i) the item prices be defined at the item level, (ii) the item prices not be different for different bidders, i.e., the price vector $\mathbf{p}$ be of the form $\mathbf{p} = \langle p^a,\ a \in L \rangle$, and ($iii$) bidder i desires no allocation different from $\mathbf{x}$, where allocation $\mathbf{x}$ solves GSWMP. Characteristic

(iii) disincentivizes after-sale arbitrage amongst the bidders, while characteristics (i) and (ii) simplify implementation in practice because prices are at the item-level and are independent of bidder characteristics. Essentially, item-level prices are straightforward to explain to bidders: *every item has a price and the price of a set of items (bundle) is simply the sum of the individual item prices.* More complex pricing schemes[16] would (i) involve pricing bundles (as opposed to individual items), or (ii) setting prices based on the bidders' identity, or (iii) a combination of both. This implies that (i) the auctioneer would require an exponentially sized price vector, making computation and communication of such prices tedious in practice, and (ii) the auctioneer would need to charge different bidders unequal prices for the same items (or bundles), which may be difficult to justify in practice. For these reasons, it is in the auctioneer's interest that the pricing structure be kept simple. We formally define the desired structure $\mathbf{p} \equiv \langle p^a,\ a \in L \rangle$ in Definition 5.1.

Definition 5.1 ***Simple and anonymous prices.*** *We refer to a vector of prices as simple and anonymous if these prices are defined for each item, rather than a bundle of items, and are not dependent on the identity of the bidders.*

Under simple and anonymous prices $\mathbf{p}$, if bidder i is assigned items in the set $S_i(\mathbf{x})$, the price he pays for the bundle $S_i(\mathbf{x})$ is the sum of the prices of the items the set contains, i.e., he pays $\sum\limits_{a \in S_i(\mathbf{x})} p^a$. His surplus is, therefore,

$$r_i(\mathbf{x}; \mathbf{p}) = \left[v_i(\mathbf{x}) - \sum_{a \in S_i(\mathbf{x})} p^a \right].$$

Iterative auctions using simple and anonymous item prices can be made to terminate at allocation $\mathbf{x}$, if $\mathbf{x}$ is such that $\cap_{i \in B} S_i(\mathbf{x}) = \varnothing$ (Candogan et. al. 2015). Then, at the termination of the iterative auction, the vector of item prices $\mathbf{p}$ and an allocation $\mathbf{x}$ constitute an equilibrium. We formalize this equilibrium notion in Definition 5.2.

Definition 5.2 ***Equilibrium with simple and anonymous prices (ESAP).*** *An allocation and price tuple* $(\mathbf{x}, \mathbf{p})$ *is an equilibrium with simple and anonymous* *pricing (ESAP) if allocation* $\mathbf{x}$ *and prices* $\mathbf{p}$ *maximizes bidder i's surplus, i.e.,* $r_i(\mathbf{x};\mathbf{p}) \geqslant r_i(\mathbf{x}';\mathbf{p}'), \quad \forall\ \mathbf{x}' \neq \mathbf{x},\ \forall\ \mathbf{p}' \neq \mathbf{p}$. *If* $(\mathbf{x},\mathbf{p})$ *is ESAP, we refer to* $\mathbf{x}$ *as ESAP allocation and to* $\mathbf{p}$ *as ESAP prices.*

Intuitively, Definition 5.2 describes a feasible allocation and a set of item prices where no bidder has the incentive to change his allocation, i.e. it disallows after-sale arbitrage opportunities. An immediate question is: Is an optimal solution to GSWMP supported by ESAP prices? Notice that ESAP may not always exist for such functions. However, if ESAP exists, it is possible to design iterative auction formats that guide item prices to ESAP prices and results in a feasible allocation that maximizes the bidders' surpluses, i.e., ESAP allocations. Consequently, the social welfare is maximized too. In this regard, it is necessary to understand the conditions on $H(\mathbf{x}) = \sum_{i\in B} v_i(\mathbf{x})$ when ESAP exists. To do so, we consider the following optimization problem, (5.5) – (5.8), obtained by relaxing the binary constraints (5.4) in GSWMP.

$$\max_{\mathbf{x}} \qquad H(\mathbf{x}) = \sum_{i\in B}\Bigg[v_i(\mathbf{x})\Bigg], \tag{5.5}$$

$$\text{subject to} \qquad \sum_{i\in B} x_i^a = 1 \quad \forall\ a \in L, \quad \cdots \quad (p^a) \tag{5.6}$$

$$x_i^a \leqslant 1 \quad \forall\ (i,a) \in BL, \quad \cdots \quad (\rho_i^a) \tag{5.7}$$

$$x_i^a \geqslant 0 \quad \forall\ (i,a) \in BL. \tag{5.8}$$

Here, $\mathbf{p} = \langle p^a,\ a \in L\rangle$ and $\rho = \langle \rho_i^a,\ (i,a) \in BL\rangle$ are the Lagrange (dual) multipliers associated with constraints (5.6) and (5.7). A feasible solution to optimization problem (5.5) – (5.8) always exists since the constraint set (5.6) – (5.8) represents a non-empty, compact, polyhedron. Note that constraint (5.7) is redundant. However, we retain this constraint in the formulation as it lends economic interpretation to the corresponding dual variables at dual optimality. This will be made clear shortly. Let $\mathbf{x}^{gopt}$ be the optimal solution to this optimization problem. Let $\mathbf{p}^{gopt}$ and ρ^{gopt} be the optimal values of the dual of this optimization problem. Let $G(\mathbf{p}^{gopt}, \rho^{gopt})$ be the optimal objective value of the dual of this optimization problem. Note that

$G(\mathbf{p}^{gopt}, \rho^{gopt}) = \sum_{a \in L} p^{a,gopt} + \sum_{(i,a) \in BL} \rho_i^{a,gopt}$. Theorem 5.1 describes the conditions on $H(\mathbf{x}) = \sum_{i \in B} v_i(\mathbf{x})$ for which ESAP exists.

Theorem 5.1 *Given the social welfare function $H(\mathbf{x})$, an ESAP exists if and only if $\mathbf{x}^{gopt}$ is integral and $H(\mathbf{x}^{gopt}) = G(\mathbf{x}^{gopt}, \mathbf{p}^{gopt})$.*

Proof of Theorem 5.1.

The first order conditions of optimality given below are satisfied when $\mathbf{x} = \mathbf{x}^{gopt}, \mathbf{p} = \mathbf{p}^{gopt}$, and $\rho = \rho^{gopt}$:

$$\left.\frac{\partial H(\mathbf{x})}{\partial x_i^a}\right|_{\mathbf{x}=\mathbf{x}^{gopt}} - p^{a,gopt} - \rho_i^{a,gopt} \leqslant 0 \quad \forall\ (i,a) \in BL, \tag{5.9}$$

$$\sum_{i \in B} x_i^{gopt} = 1 \quad \forall\ a \in L, \tag{5.10}$$

$$\rho_i^{a,gopt}(1 - x_i^{a,gopt}) = 0 \quad \forall\ (i,a) \in BL, \tag{5.11}$$

$$\rho_i^{a,gopt} \geqslant 0 \quad \forall\ (i,a) \in BL. \tag{5.12}$$

The dual formulation of (5.5) – (5.8) is as follows:

$$\min \qquad \sum_{a \in L} p^a + \sum_{(i,a) \in BL} \rho_i^a, \tag{5.13}$$

$$\text{subject to} \qquad \left.\frac{\partial H(\mathbf{x})}{\partial x_i^a}\right|_{\mathbf{x}=\mathbf{x}^{gopt}} - p^a - \rho_i^a \leqslant 0 \quad \forall\ (i,a) \in BL, \tag{5.14}$$

$$\rho_i^a(1 - x_i^{a,gopt}) = 0 \quad \forall\ (i,a) \in BL, \tag{5.15}$$

$$\rho_i^a \geqslant 0 \quad \forall\ (i,a) \in BL. \tag{5.16}$$

($\Longrightarrow$) *Suppose $\mathbf{x}^{gopt}$ is integral and $H(\mathbf{x}^{gopt}) = G(\mathbf{p}^{gopt}, \rho^{gopt})$. We now show that the values of $\mathbf{p}^{gopt}$ and ρ^{gopt} satisfying* (5.9) – (5.12) *and $\mathbf{x}^{opt}$ correspond to an ESAP.*

Since $H(\mathbf{x}^{opt}) = G(\mathbf{p}^{gopt}, \rho^{gopt})$ and $\mathbf{x}^{gopt}$ is integral, we have that $\rho_i^{a,gopt} = \frac{\partial H(\mathbf{x})}{\partial x_i^a}\big|_{\mathbf{x}=\mathbf{x}^{gopt}} - p^{a,gopt}$ if $x_i^{a,gopt} = 1$, and $\rho_i^{a,gopt} = 0$ if $x_i^{a,gopt} = 0$. The set $S_i(\mathbf{x}^{gopt}) = \{a : x_i^{a,gopt} =$

$1, \quad a \in L\}$ for all $i \in B$. Thus,

$$\sum_{a \in S_i(\mathbf{x}^{gopt})} \rho_i^{a,gopt} = v_i(\mathbf{x}^{gopt}) - \sum_{a \in S_i(\mathbf{x}^{gopt})} p^{a,gopt},$$

since

$$\sum_{a \in S_i(\mathbf{x}^{gopt})} \left. \frac{\partial H \mathbf{x})}{\partial x_i^a} \right|_{\mathbf{x}=\mathbf{x}^{gopt}} = v_i(\mathbf{x}^{gopt}).$$

Now, $r_i(\mathbf{x}^{gopt}; \mathbf{p}^{gopt}) = v_i(\mathbf{x}^{gopt}) - \sum_{i \in B} p^{a,gopt}$ is bidder i's maximum surplus, since in the dual formulation, we minimize ρ_i^a $(i,a) \in BL$ with lower bounds on ρ_i^a, $(i,a) \in BL$, and we have that $p^{a,gopt} > 0$ for all $a \in L$, where

$$p^{a,gopt} = \text{second-highest}_{k \in B} \left. \frac{\partial H(\mathbf{x})}{\partial x_k^a} \right|_{\mathbf{x}=\mathbf{x}^{gopt}}, \ \forall \ a \in L$$

and $\mathbf{x}^{gopt}$ is feasible by definition. Thus, $(\mathbf{x}^{opt}, \mathbf{p}^{opt})$ is ESAP.

$(\Longleftarrow)$ *Suppose ESAP exists. We now show that* $\mathbf{x}^{gopt}$ *is integral and* $H(\mathbf{x}^{gopt}) = G(\mathbf{p}^{gopt}, \rho^{gopt})$.

With this supposition, $\mathbf{x}^{gopt}$ is integral, since $\mathbf{x}^{gopt}$ must be an undivided allocation of items and is such that $\sum_{i \in B} x_i^{a,gopt} = 1$ for all $a \in L$, and $(\mathbf{x}^{gopt}, \mathbf{p}^{gopt})$ is such that bidder i's surplus is maximized (from Definition 5.2), i.e.,

$$v_i(\mathbf{x}^{gopt}) - \sum_{a \in S_i(\mathbf{x}^{gopt})} p^{a,gopt} = \max_{\mathbf{x}} \Big[v_i(\mathbf{x}) - \sum_{a \in S_i(\mathbf{x})} p^{a,gopt} \Big],$$

where

$$p^{a,gopt} = \text{second-highest}_{k \in B} \left. \frac{\partial H(\mathbf{x})}{\partial x_k^a} \right|_{\mathbf{x}=\mathbf{x}^{gopt}}, \ \forall \ a \in L.$$

Thus, $\mathbf{x}^{gopt}$ is integral. Now,

$$G(\mathbf{p}^{gopt}, \rho^{gopt}) = \min \qquad \sum_{a \in L} p^a + \sum_{(i,a) \in BL} \rho_i^a,$$

$$\text{subject to} \qquad \frac{\partial H(\mathbf{x})}{\partial x_i^a}\Big|_{\mathbf{x}=\mathbf{x}^{gopt}} - p^a - \rho_i^a \leqslant 0 \quad \forall\ (i,a) \in BL,$$

$$\rho_i^a(1 - x_i^{a,gopt}) = 0 \quad \forall\ (i,a) \in BL,$$

$$\rho_i^a \geqslant 0 \quad \forall\ (i,a) \in BL.$$

The constraints follow from the first order conditions of optimality when $\mathbf{x} = \mathbf{x}^{gopt}$. This is, therefore, a linear program. We have that

$$\rho_i^{a,gopt} = \max\{0, \frac{\partial H(\mathbf{x})}{\partial x_i^a}\big|_{\mathbf{x}=\mathbf{x}^{gopt}} - p^{a,gopt}\}$$

and

$$p^{a,gopt} = \text{second-highest}_{k\in B} \quad \frac{\partial H(\mathbf{x})}{\partial x_k^a}\Bigg|_{\mathbf{x}=\mathbf{x}^{gopt}}, \ \forall\ a \in L.$$

Here, for all $a \in L$, we have that $\rho_i^{a,gopt} > 0$ only for one i in the set B, and $x_i^{a,gopt} = 1$ for this (i,a). Thus,

$$\sum_{a\in S_i(\mathbf{x}^{gopt})} \rho_i^{a,gopt} = \sum_{a\in S_i(\mathbf{x}^{gopt})} \frac{\partial H(\mathbf{x})}{\partial x_i^a}\big|_{\mathbf{x}=\mathbf{x}^{gopt}} - \sum_{a\in S_i(\mathbf{x}^{gopt})} p^{a,gopt}$$

$$= v_i(\mathbf{x}^{gopt}) - \sum_{a\in S_i(\mathbf{x}^{gopt})} p^{a,gopt}.$$

Summing across all bidders,

$$\sum_{i\in B}\sum_{a\in S_i(\mathbf{x}^{gopt})} \rho_i^{a,gopt} = \sum_{i\in B} v_i(\mathbf{x}^{gopt}) - \sum_{i\in B}\sum_{a\in S_i(\mathbf{x}^{gopt})} p^{a,gopt}$$

$$= \sum_{i\in B} v_i(\mathbf{x}^{gopt}) - \sum_{a\in L} p^{a,gopt}$$

$$\implies \sum_{(i,a)\in BL} \rho_i^{a,gopt} = \sum_{i\in B} v_i(\mathbf{x}^{gopt}) - \sum_{a\in L} p^{a,gopt}$$

$$\implies \sum_{(i,a)\in BL} \rho_i^{a,gopt} + \sum_{a\in L} p^{a,gopt} = \sum_{i\in B} v_i(\mathbf{x}^{gopt})$$

Therefore, we have that

$$G(\mathbf{p}^{gopt}, \rho^{gopt}) = \sum_{a \in L} p^{a,gopt} + \sum_{(i,a) \in BL} \rho_i^{a,gopt} = \sum_{i \in B} v_i(\mathbf{x}^{gopt}) = H(\mathbf{x}^{gopt})$$

Thus, we prove Theorem 5.1. ▫

If $\mathbf{x}^{gopt}$ is integral, $\mathbf{x}^{gopt}$ also solves GSWMP, since $\mathbf{x}^{gopt}$ is feasible to GSWMP. Thus, when $\mathbf{x}^{gopt}$ is integral, the optimal solution to the dual, i.e., $\left\langle \rho_i^{a,gopt},\ (i,a) \in BL \right\rangle$ and $\langle p^{a,gopt},\ a \in L\rangle$ has the following economic interpretation: the quantity $\sum_{a \in S_i(\mathbf{x}^{gopt})} \rho_i^{a,gopt}$ represents bidder i's surplus from allocation $\mathbf{x}^{gopt}$, i.e., $r_i(\mathbf{x}^{gopt}; \mathbf{p}^{gopt}) = \sum_{a \in S_i(\mathbf{x}^{gopt})} \rho_i^{a,gopt}$, and the quantity $p^{a,gopt}$ represents the item prices. Since the dual of (5.5) – (5.8) has multiple optima, the value of $p^{a,gopt},\ a \in L$ can be chosen in such a way it that can be made as low as possible without affecting the item allocation $\mathbf{x}^{gopt}$. Therefore, the components $\langle p^{a,gopt},\ a \in L\rangle$ of the optimal dual solution to (5.5) – (5.8) represent the prices of the items needed to support an allocation $\mathbf{x}^{gopt}$ that maximizes the bidders' social welfare. On the other hand, if $\mathbf{x}^{gopt}$ is not integral, the solution $\langle p^{a,gopt},\ a \in L\rangle$ *do not* represent prices that support a social-welfare-maximizing allocation, since $\mathbf{x}^{gopt}$ is *not feasible* to the social welfare maximization problem GSWMP. Thus, Theorem 5.1 summarizes the necessary and sufficient conditions for ESAP to exist.

5.3.2 Pairwise-Additive Negative Value Externalities (PANE)

In this subsection, we define the PANE valuation function and describe how it models negative externalities in our context. Let $\mathbf{x}$ be an allocation. Under PANE, bidder i's valuation $v_i(\mathbf{x})$ for allocation $\mathbf{x}$ is as follows:

$$v_i(\mathbf{x}) \equiv \sum_{a \in L} u_i^a x_i^a - \sum_{a \in L} \sum_{\substack{(j,b) \in BL \\ j \neq i \\ b \neq a}} w_{ij}^{ab} x_i^a x_j^b. \tag{5.17}$$

Here, the term u_i^a, where $u_i^a > 0,\ (i,a) \forall \in BL$, is bidder i's private value for item a if no other bidder $j \neq i,\ j \in B$ is assigned any other item. Bidder i's value for item a reduces by an amount w_{ij}^{ab}, where $w_{ij}^{ab} \geqslant 0$, if bidder $j \neq b$ is assigned item $b \neq a$. Here, we assume that $w_{ij}^{ab} = 0$ if $i = j$ or if $a = b$. This is because a bidder does not impose an externality on himself. Further, since an item can only be assigned to one bidder, we assume that $w_{ij}^{aa} = 0$. Essentially, w_{ij}^{ab} captures the pairwise-additive negative value externalities that bidder i faces for item a when bidder $j \neq i$ is assigned item $b \neq a$. As mentioned earlier, $w_{ij}^{ab},\ (i,a) \in BL,\ (j,b) \in BL$ are private information for bidder i. Under PANE, $x_i^a = 1$ if $a \in S_i(\mathbf{x})$ and $x_i^a = 0$ if $a \notin S_i(\mathbf{x})$, and $x_j^b = 1$ for all $(j,b) \in BL,\ j \neq i,\ b \neq a$ if $b \in S_j(\mathbf{x})$ and $x_j^b = 0$ if $b \notin S_j(\mathbf{x})$.

We now illustrate how PANE may be interpreted in practice: For example, assigning w_{ij}^{ab} a large value, say M where $M >> u_i^a$, is equivalent to bidder i having the following preference: bidder i values item a only if bidder j is not assigned item b. Therefore, if $w_{ij}^{ab} > u_i^a$ for any $j \neq i$ and $b \neq a$, it means that bidder i does not consider item a valuable if item b is assigned to bidder j. If $w_{ij}^{ab} > u_i^a$ for more than one j where $j \neq i$, it implies that bidder i does not consider item a valuable if any of such j's receive the item. If, for some allocation $\mathbf{x}$, we have

$$u_i^a - \sum_{\substack{(j,b)\in BL \\ j\neq i \\ b\neq a}} w_{ij}^{ab} x_j^b > 0,$$

it means that bidder i considers item a valuable, but at a lower value than u_i^a. If $w_{ij}^{ab} = 0$, bidder i's value for item a is indifferent to the assignment of item b to bidder j. Notice that the externality parameter w_{ij}^{ab} is both item-specific *and* bidder-specific.

In the context of online advertisement slots. Bidder i can specify a set of bidders B_0 he does not want taking a specific set of slots L_0 by setting $w_{ij}^{ab} = M$ for all $j \in B_0 \subseteq B$ and $a \in L_0 \subseteq L$ for some or all $a \in L$. If slot a is located above slot b, and if bidder i values slot a at a positive value only if a bidder $j \in B$ is *not* assigned slot b, $w_{ij}^{ab} = M$. If bidder i values a slot a positively only when no other bidder $j \in B$

is assigned a slot, say c, that is higher than a, then $w_{ij}^{ac} = M$ for all $j \in B,\ j \neq i$. These are examples of preferences where a bidder does not want to share the same webpage with other advertisers. On the other hand, setting values of $w_{ij}^{ab} < M$ implies that bidder i is open to sharing a webpage with other advertisers, although he would not value his slots as much as he would have without such an allocation. Thus, the PANE valuation function captures some common forms of valuations in the presence of allocation externalities discussed in the literature (Bhargava et. al. 2019, Constantin et. al. 2011, Sayedi et. al. 2018).

Next, we discuss whether ESAP exists under PANE.

5.4 PANE and Solving for the Social-Welfare-Maximizing Allocation

In this section, we discuss structural results related to PANE and show that ESAP exists under the PANE valuation. This result has implications for the design of iterative auctions that terminate at a social-welfare-maximizing allocation with simple and anonymous prices. As explained in §5.1, iterative auctions modify prices based on the interest shown by the bidders for the items for a given set of prices. If the prices of the items are simple and anonymous, the process of modifying the item prices across iterations will not be computationally difficult for the auctioneer, since she only has to modify $|L|$ prices every iteration. This is in contrast to modifying an exponential number of prices, which would be the case if the price structure needed to achieve a social-welfare-maximizing allocation was neither simple nor anonymous.

In §5.4.1, we begin by describing the social welfare maximization problem under PANE. This problem, that we call QIP, is an instance of GSWMP where $H(\mathbf{x})$ is the PANE valuation function. Specifically, we show that the optimization problem obtained by relaxing the binary constraints on the variables $\mathbf{x}$, that we call QP, continues to yield integral optimal solutions (integrality property). Next, we show that problem QP satisfies the strong duality property, i.e., no gap exists between the primal and dual objective values at optimality. Finally, we show that Theorem 5.1 immediately follows by which we establish the existence of ESAP under PANE. The results established in §5.4.1 are also of interest from a computational tractability standpoint. In §5.4.2, we establish that the computational complexity of PANE is bounded polynomially. This is a result of interest since a direct mechanism in the form of a VCG mechanism can achieve a social-welfare-maximizing allocation.

5.4.1 Integrality Property and Strong Duality

We first examine the problem of finding a social-welfare-maximizing allocation of items to bidders. Note that $w_{ij}^{ab} = 0$ if $i = j$ or $a = b$. We call this problem QIP:

$$\textbf{QIP:} \qquad \max_{\mathbf{x}} \sum_{(i,a)\in BL} u_i^a x_i^a - \sum_{(i,a)\in BL} \sum_{\substack{(j,b)\in BL \\ j\neq i \\ b\neq a}} w_{ij}^{ab} x_i^a x_j^b, \tag{5.18}$$

$$\text{subject to} \qquad \sum_{i\in B} x_i^a = 1 \quad \forall\ a \in L, \tag{5.19}$$

$$x_i^a \in \{0,1\} \quad \forall\ (i,a) \in BL. \tag{5.20}$$

Problem QIP finds the allocation that maximizes social welfare resulting from the allocation.

The integrality property. Consider the continuous relaxation of QIP, denoted as QP.

$$\textbf{QP:} \qquad \max_{\mathbf{x}} \sum_{(i,a)\in BL} u_i^a x_i^a - \sum_{(i,a)\in BL} \sum_{\substack{(j,b)\in BL \\ j\neq i \\ b\neq a}} w_{ij}^{ab} x_i^a x_j^b, \tag{5.21}$$

$$\text{subject to} \qquad \sum_{i\in B} x_i^a = 1 \quad \forall a \in L, \quad \cdots (p^a) \tag{5.22}$$

$$x_i^a \leqslant 1 \quad \forall\ (i,a) \in BL, \quad \cdots (\rho_i^a) \tag{5.23}$$

$$x_i^a \geqslant 0 \quad \forall\ (i,a) \in BL. \tag{5.24}$$

For notational ease, we use the following matrices to represent the problem parameters.

1. $\mathbf{u}$ is a column vector of dimensions $|BL| \times 1$ whose $(i,a)^{th}$ component is u_i^a.
2. $\mathbf{W}$ is a matrix of dimension $|BL| \times |BL|$ whose $[(i,a),(j,b)]^{th}$ component is w_{ij}^{ab}.
3. $\mathbf{A}$ is the coefficient matrix of constraint (5.22)
4. $\mathbf{x}$ is a column vector of dimensions $|BL| \times 1$ whose $(i,a)^{th}$ component is x_i^a.
5. $\mathbf{p}$ is a column vector of dimension $|L| \times 1$ whose a^{th} component is p^a.
6. ρ is a column vector of dimensions $|BL| \times 1$ whose $(i,a)^{th}$ component is ρ_i^a.

In problem QP, $p^a,\ a \in L$ and $\rho_i^a,\ (i,a) \in BL$ denote the multipliers associated with the constraints (5.22) and (5.23). We keep constraint (5.23) despite its redundancy because it allows for an economic interpretation of its multipliers $\rho_i^a,\ (i,a) \in BL$. In general, the objective function (5.21) of QP is neither convex nor concave because the matrix $\mathbf{W}$ may be indefinite. In Theorem 5.2, we show that the QP yields integral solutions.

Theorem 5.2 *(Integrality property of QP) The solutions to problem QP are integral, i.e., the values of $x_i^a,\ (i,a) \in BL$ at optimality of QP are either 0 or 1.*

Proof of Theorem 5.2. The proof of this theorem has already been presented in the proof of Theorem 3.1.

From a technical standpoint, the fact that $w_{ij}^{ab} = 0$ when $i = j$ or $a = b$ plays a role in the integrality property of QP. If $w_{ij}^{ab} > 0$ when $i = j$ or $a = b$, the integrality property of QP may not hold in general. Next, we establish strong duality for QP.

Strong duality. The Lagrangean relaxation of QP, $QP\text{-}L(\mathbf{x};\mathbf{p},\rho)$, is as below.

$$QP\text{-}L(\mathbf{x};\mathbf{p},\rho) = \sum_{(i,a)\in BL} u_i^a x_i^a - \sum_{(i,a)\in BL} \sum_{\substack{(j,b)\in BL \\ j\neq i \\ b\neq a}} w_{ij}^{ab} x_i^a x_j^b + \sum_{a\in L} p^a \Big[1 - \sum_{i\in B} x_i^a\Big] + \sum_{(i,a)\in BL} \rho_i^a \Big[1 - x_i^a\Big]. \tag{5.25}$$

In matrix notation, $QP\text{-}L(\mathbf{x};\mathbf{p},\rho)$ can be written as

$$QP\text{-}L(\mathbf{x};\mathbf{p},\rho) = \mathbf{p}^T\mathbf{1} + \rho^T\mathbf{1} + \mathbf{u}^T\mathbf{x} - \mathbf{p}^T\mathbf{A}\mathbf{x} - \rho^T\mathbf{x} - \mathbf{x}^T\mathbf{W}\mathbf{x}, \tag{5.26}$$

where $\mathbf{1}$ is a column vector all of whose components are 1. Here, $\mathbf{x} \geqslant 0$, since we are only considering the non-negative orthant. The first-order conditions for optimality of QP (labelled FOOC) are as follows.

$$\textbf{FOOC:} \qquad u_i^a - \sum_{\substack{(j,b)\in BL \\ j\neq i \\ b\neq a}} w_{ij}^{ab} x_j^b - p^a - \rho_i^a \leqslant 0 \quad \forall\ (i,a) \in BL, \tag{5.27}$$

$$-\sum_{i\in B} x_i^a + 1 = 0 \quad \forall\ a \in L, \tag{5.28}$$

$$\rho_i^a \geqslant 0 \quad \forall\ (i,a) \in BL, \tag{5.29}$$

$$\rho_i^a(1 - x_i^a) = 0 \quad \forall\ (i,a) \in BL. \tag{5.30}$$

Conditions (5.27), (5.28), and (5.29) are obtained by taking partial derivatives of $QP\text{-}L(\mathbf{x};\mathbf{p},\rho)$ with respect to (w.r.t) to the components of $\mathbf{x},\mathbf{p},\rho$. Condition (5.29) is the condition for dual feasibility, and conditions (5.30) are the complementary slackness conditions. Let Ω be the set of values for $\mathbf{x}$ that satisfy FOOC.

Theorem 5.3 *(Strong duality) Let $\mathbf{x}^*$ be the optimal solution to QP. Let $(\mathbf{p}^*,\rho^*)$ represent the optimal solution to the dual of QP. Let QP-P($\mathbf{x}^*$) represent the optimal value of QP at $\mathbf{x}^*$. Let QP-D represent the optimal value of the dual of QP evaluated at $(\mathbf{p}^*,\rho^*)$. Then QP-P$(\mathbf{x}^*)$ = QP-D$(\mathbf{p}^*,\rho^*)$.*

Proof of Theorem 5.3.

Consider a locally-optimal solution $\bar{\mathbf{x}} \in \Omega$. Given $\bar{\mathbf{x}}$, let the solution to variables $\mathbf{p}$ and ρ from solving FOOC be $\mathbf{p}(\bar{\mathbf{x}})$ and $\rho(\bar{\mathbf{x}})$. If $\bar{\mathbf{x}} = \mathbf{x}^*$, we have that $\mathbf{p}(\bar{\mathbf{x}} = \mathbf{x}^*) = \mathbf{p}^*$ and $\rho(\bar{\mathbf{x}} = \mathbf{x}^*) = \rho^*$. We show this below. The values of $\mathbf{p}(\bar{\mathbf{x}})$ and $\rho(\bar{\mathbf{x}})$ can be obtained as a solution to the following linear programming problem (it is a linear program since $\bar{\mathbf{x}}$ is fixed):

$$\min_{\substack{p^a,\rho_i^a \\ i\in B \\ a\in L}} \quad \sum_{a\in L} p^a + \sum_{(i,a)\in BL} \rho_i^a, \tag{5.31}$$

$$\text{subject to} \quad u_i^a - \sum_{(j,b)\in BL} w_{ij}^{ab}\bar{x}_j^b \leqslant p^a + \rho_i^a \quad \forall\ (i,a) \in BL, \tag{5.32}$$

$$-\sum_{i\in B} \bar{x}_i^a + 1 = 0 \quad \forall\ a \in L, \tag{5.33}$$

$$\rho_i^a \geqslant 0 \quad \forall\ (i,a) \in BL, \tag{5.34}$$

$$\rho_i^a(1-\bar{x}_i^a) = 0 \quad \forall\ (i,a) \in BL. \tag{5.35}$$

The solution $\mathbf{p}(\bar{\mathbf{x}})$ and $\rho(\bar{\mathbf{x}})$ are as follows:

$$\rho_i^a(\bar{\mathbf{x}}) = \begin{cases} u_i^a - \sum_{(j,b)\in BL} w_{ij}^{ab}\bar{x}_j^b - p^{a*}(\bar{\mathbf{x}}),\ \text{if } i = \arg\max_{k\in B} u_k^a - \sum_{(k,b)\in BL} w_{ik}^{ab}\bar{x}_k^b - p^{a*}(\bar{\mathbf{x}}),\ \text{and} \\ \max_{k\in B} u_k^a - \sum_{(k,b)\in BL} w_{ik}^{ab}\bar{x}_k^b - p^{a*}(\bar{\mathbf{x}}) > 0, \\ \\ 0 \text{ otherwise,} \end{cases}$$

for all $(i,a) \in BL$ and

$$p^{a*}(\bar{\mathbf{x}}) = \text{ second-highest }_{k\in B} \left[u_k^a - \sum_{(k,b)\in BL} w_{ik}^{ab}\bar{x}_k^b\right],$$

for all $a \in L$. We point out that the solutions to $\mathbf{p}(\bar{x})$ and $\rho(\bar{x})$ are infinite in number. We only present one of these solutions. A key point to note here is that for each $a \in L$, $\rho_k^a(\bar{\mathbf{x}}) > 0$ only for one $k \in B$. Let $B^*(\bar{\mathbf{x}}) = \{k : k \in B, \rho_k^a(\bar{\mathbf{x}}) > 0 \quad \forall a \in L\}$. Thus, from the condition $\rho_i^a(1-\bar{x}_i^a) = 0$ for all $(i,a) \in L$, we have that $\bar{x}_k^a = 1$ if $\rho_k^a > 0$. Consider the expression

$$\sum_{a\in L} p^{a*}(\bar{\mathbf{x}}) + \sum_{(i,a)\in BL} \rho_i^a(\bar{\mathbf{x}}). \tag{5.36}$$

We have that

$$\sum_{a\in L} p^{a*}(\bar{\mathbf{x}}) + \sum_{(i,a)\in BL} \rho_i^a(\bar{\mathbf{x}}) = \sum_{a\in L} p^{a*}(\bar{\mathbf{x}}) + \sum_{a\in L}\sum_{i\in B^*(\bar{\mathbf{x}})} \rho_i^a(\bar{\mathbf{x}}) + \sum_{a\in L}\sum_{i\in B-B^*(\bar{\mathbf{x}})} \rho_i^a(\bar{\mathbf{x}})$$

$$= \sum_{a\in L}\sum_{i\in B^*(\bar{\mathbf{x}})} \Big[u_i^a - \sum_{(j,b)\in B^*(\bar{\mathbf{x}})} w_{ij}^{ab}\bar{x}_j^{b*}\Big] + 0$$

$$= \sum_{a\in L}\sum_{i\in B^*(\bar{\mathbf{x}})} \Big[u_i^a - \sum_{(j,b)\in B^*(\bar{\mathbf{x}})} w_{ij}^{ab}\bar{x}_j^{b*}\Big]\bar{x}_i^{a*}.$$

since $\bar{x}_i^a = 1$ if $(i,a) \in B^*(\bar{\mathbf{x}})$. Now, consider the solution $\mathbf{y}^*$ where

$$\mathbf{y}^* = \arg\max_{\bar{\mathbf{x}}\in\Omega} \left[\sum_{a\in L} p^{a*}(\bar{\mathbf{x}}) + \sum_{(i,a)\in BL} \rho_i^a(\bar{\mathbf{x}})\right] = \arg\max_{\bar{\mathbf{x}}\in\Omega} \sum_{a\in L}\sum_{i\in B^*(\bar{\mathbf{x}})} \Big[u_i^a - \sum_{(j,b)\in B^*(\bar{\mathbf{x}})} w_{ij}^{ab}\bar{x}_j^{b*}\Big]\bar{x}_i^{a*}. \tag{5.37}$$

Thus, solving the maximization problem (5.37) is the same as solving problem QP to global optimality. When $\bar{\mathbf{x}} = \mathbf{y}^*$, we have that

$$\sum_{a\in L} p^{a*}(\mathbf{y}^*) + \sum_{a\in L}\sum_{i\in B} \rho_i^a(\mathbf{y}^*) = \sum_{a\in L} p^{a*}(\mathbf{y}^*) + \sum_{a\in L}\sum_{i\in B^*(\mathbf{y}^*)} \rho_i^a(\mathbf{y}^*) + \sum_{a\in L}\sum_{i\in B-B^*(\mathbf{y}^*)} \rho_i^a(\mathbf{y}^*)$$

$$= \sum_{a\in L}\sum_{i\in B^*(\mathbf{y}^*)} \Big[u_i^a - \sum_{(j,b)\in B^*(\mathbf{y}^*)L} w_{ij}^{ab}y_j^{b*}\Big] + 0$$

$$= \sum_{a\in L}\sum_{i\in B^*(\mathbf{y}^*)} \Big[u_i^a - \sum_{(j,b)\in B^*(\mathbf{y}^*)L} w_{ij}^{ab}y_j^{b*}\Big]y_i^{a*} = \mathit{QP\text{-}P}(\mathbf{x}^*),$$

since $y_i^{a*} = 1$ for all $i \in B^*(\mathbf{y}^*)$ and for all $a \in L$. Thus, $\mathbf{y}^*$ is a global maximizer of QP.

Let the dual variables associated with constraints (5.22) and (5.23) be $q^a, a \in L$ and $\phi_i^a, (i,a) \in BL$. The expression for $\mathit{QP\text{-}L}(\mathbf{x};\mathbf{q},\phi)$ can be written as

$$\mathit{QP\text{-}L}(\mathbf{x};\mathbf{q},\phi) = \phi^T\mathbf{1} + \mathbf{q}^T\mathbf{1} + \Big[\mathbf{u}^T - \mathbf{q}^T\mathbf{A} - \phi^T - \mathbf{x}^T\mathbf{W}\Big]\mathbf{x}. \tag{5.38}$$

Here $\mathbf{1}$ is a column vector each of whose components is 1. Let $\mathbf{x} = \mathbf{y}^*$ (note that $\mathbf{y}^*$

is the global optimizer of QP as defined in (5.37)). Finding the value of the variables $\mathbf{q}$ and ϕ such that $QP\text{-}L(\mathbf{x} = \mathbf{y}^*; \mathbf{q}, \phi)$ is minimized requires that

$$\mathbf{u}^T - \mathbf{q}^T\mathbf{A} - \phi^T - \mathbf{y}^{*T}\mathbf{W} \leqslant \mathbf{0}^T, \qquad \phi \geqslant \mathbf{0},$$

where $\mathbf{0}$ is a column vector each of whose components is 0. The values of $\mathbf{q}^*$ and ϕ^* can be computed by solving the following optimization problem:

$$(\mathbf{q}^*, \phi^*) = \arg\min_{\mathbf{q},\phi} QP\text{-}L(\mathbf{y}^*; \mathbf{q}, \phi) \tag{5.39}$$

$$\text{subject to} \qquad \mathbf{u}^T - \mathbf{q}^T\mathbf{A} - \phi^T - \mathbf{y}^{*T}\mathbf{W} \leqslant \mathbf{0}^T, \tag{5.40}$$

$$\phi \geqslant \mathbf{0}. \tag{5.41}$$

Thus, the dual of QP can be formulated as

$$\min_{\mathbf{q};\phi} \mathbf{q}^T\mathbf{1} + \phi^T\mathbf{1} \tag{5.42}$$

$$\text{subject to} \qquad \mathbf{u}^T - \mathbf{q}^T\mathbf{A} - \phi^T - \mathbf{y}^{*T}\mathbf{W} \leqslant \mathbf{0}^T, \tag{5.43}$$

$$\phi \geqslant \mathbf{0}. \tag{5.44}$$

The solution $\mathbf{q}^*$ and ϕ^* is as follows:

$$\phi_i^{a*} = \begin{cases} u_i^a - \sum_{(j,b)\in BL} w_{ij}^{ab} y_j^{b*} - q^{a*}, \text{ if } i = \arg\max_{k\in B} u_k^a - \sum_{(k,b)\in BL} w_{ik}^{ab} y_k^{b*} - q^{a*}, \text{ and} \\ \max_{k\in B} u_k^a - \sum_{(k,b)\in BL} w_{ik}^{ab} y_k^{b*} - q^{a*} > 0, \\ \\ 0 \text{ otherwise.} \end{cases}$$

for all $(i, a) \in BL$ and

$$q^{a*} = \text{ second-highest }_{k\in B} \left[u_k^a - \sum_{(k,b)\in BL} w_{ik}^{ab} y_k^{b*} \right].$$

Let $P^* = \{k : k \in B, \phi_k^{a*} > 0 \quad \forall a \in L\}$. We point out that there are alternate optima yielding the same objective function value. To see how $(\mathbf{q}^*, \phi^*)$ is the solution, we substitute $(\mathbf{q}^*, \phi^*)$ in the objective expression (5.42) and obtain the following expression:

$$\sum_{a \in L} \sum_{i \in B} \Big[u_i^a - \sum_{(j,b) \in BL} w_{ij}^{ab} y_j^{b*} \Big] y_i^{a*}$$

$$= \sum_{a \in L} \sum_{i \in P^*} \Big[u_i^a - \sum_{(j,b) \in P^*L} w_{ij}^{ab} y_j^{b*} \Big] y_i^{a*} + \sum_{a \in L} \sum_{i \in B - P^*} \Big[u_i^a - \sum_{(j,b) \in B - P^*L} w_{ij}^{ab} y_j^{b*} \Big] y_i^{a*}$$

$$= \sum_{a \in L} \sum_{i \in P^*} \Big[u_i^a - \sum_{(j,b) \in P^*L} w_{ij}^{ab} y_j^{b*} \Big] + 0$$

(since $y_i^{a*} = 0$ if $(i,a) \in B - P^*$, and $y_i^{a*} = 1$ if $(i,a) \in P^*$).

$$= \sum_{a \in L} \sum_{i \in P^*} \Big[u_i^a - \sum_{(j,b) \in P^*} w_{ij}^{ab} y_j^{b*} \Big] y_i^{a*}.$$

(since $y_i^{a*} = 1$ if $i \in P^*$).

Notice that

$$\sum_{a \in L} \sum_{i \in P^*} \Big[u_i^a - \sum_{(j,b) \in P^*} w_{ij}^{ab} y_j^{b*} \Big] y_i^{a*} = \max_{\bar{\mathbf{x}} \in \Omega} \left[\sum_{a \in L} p^{a*}(\bar{\mathbf{x}}) + \sum_{(i,a) \in BL} \rho_i^a(\bar{\mathbf{x}}) \right] = QP\text{-}P(\mathbf{x}^*).$$

Thus, we have that $q^{a*} = p^a(\mathbf{y}^*) = p^a(\mathbf{x}^*)$ and $\phi_i^{a*} = \rho_i^a(\mathbf{y}^*) = \rho_i^a(\mathbf{x}^*)$. Therefore,

$$\sum_{a \in L} q^{a*} + \sum_{(i,a) \in BL} \phi_i^{a*} = QP\text{-}D(\mathbf{p}^*, \rho^*) = \max_{\bar{\mathbf{x}} \in \Omega} \left[\sum_{a \in L} p^{a*}(\bar{\mathbf{x}}) + \sum_{(i,a) \in BL} \rho_i^a(\bar{\mathbf{x}}) \right] = QP\text{-}P(\mathbf{x}^*).$$

Thus, we prove Theorem 5.3. These results also prove Corollary 5.1. □

In general, the duality gap for nonconvex quadratic optimization problems is non-zero (Boyd et. al. 1996, 2004). However, formulation QP is different in that regard because semi-assignment constraints (5.22) play an important role in ensuring that the duality gap is zero at optimality. In particular, constraints (5.22) are separable in $a,\ a \in L$. As we establish later, this structural property of strong duality property has important implications for computational tractability. It is also noteworthy that QP demonstrates total dual integrality if the entries of matrices $\mathbf{u}$ and $\mathbf{W}$ are integral. We summarize this result in Corollary 5.1.

Corollary 5.1 *(Corollary to Theorem 5.3: Dual integrality) If the components of* $\mathbf{u}$ *and* $\mathbf{W}$ *are integer-valued, the optimal dual solution of problem QP, i.e., the components of* $\mathbf{p}^*$ *and* ρ^**, are integer-valued.*

Thus, Theorems 5.2 and 5.3 show the existence of simple and anonymous market-clearing price vectors $\langle p^a,\ a \in L\rangle$ that result in an undivided allocation $\mathbf{x}$ where every bidder i makes a surplus of $\rho_i^a \geqslant 0,\ a \in L$. Formally, we state the existence of ESAP under PANE in Theorem 5.4.

Theorem 5.4 *If the bidders' valuation functions are PANE, ESAP exists.*

Theorems 5.2 and 5.3 are of interest from a computational tractability standpoint. Optimizing functions that are neither convex nor concave is known to be NP-Hard (Zheng et. al. 2012, Pardalos et. al. 1991, Boyd et. al. 1996). To this end, in §5.4.2, we show that the problem QIP can be solved in polynomial time. Thus, we have a class of quadratic binary optimization problems that can be solved in polynomial time.

5.4.2 Computational Complexity of QIP

In this section, we show that problem QIP can be solved in polynomial time. Theorem 5.5 formalizes this result. As we shall show, this has implications on the practicality of using a VCG auction as a direct mechanism.

Theorem 5.5 *QIP can be solved in polynomial time.*

Proof of Theorem 5.5.

The key to this result is Theorem 5.3. The relaxation of problem QIP in matrix form can be written as follows with a minimization objective:

$$\min \qquad \mathbf{x}^T\mathbf{W}\mathbf{x} - \mathbf{u}^T\mathbf{x}, \tag{5.45}$$

$$\text{subject to} \qquad \mathbf{A}\mathbf{x} = \mathbf{1} \quad \cdots \quad (\lambda), \tag{5.46}$$

$$\mathbf{x} \geqslant \mathbf{0} \quad \cdots \quad (\psi). \tag{5.47}$$

Here $\lambda = \langle \lambda^a, a \in L \rangle$ and $\psi = \langle \psi_i^a, (i,a) \in BL \rangle$ are multipliers associated with constraints (5.46) and (5.47) respectively. Note that $\mathbf{x}^T\mathbf{W}\mathbf{x} = Tr(\mathbf{W}\mathbf{x}\mathbf{x}^T)$, where $Tr(\mathbf{W}\mathbf{x}\mathbf{x}^T)$ is the trace of the matrix $\mathbf{W}\mathbf{x}\mathbf{x}^T$. Thus, we rewrite the above optimization problem as follows:

$$\min \qquad Tr(\mathbf{W}\mathbf{X}) - \mathbf{u}^T\mathbf{x}, \tag{5.48}$$

$$\text{subject to} \qquad \mathbf{A}\mathbf{x} = \mathbf{1}, \tag{5.49}$$

$$\mathbf{X} = \mathbf{x}\mathbf{x}^T, \tag{5.50}$$

$$\mathbf{x} \geqslant \mathbf{0}, \tag{5.51}$$

where $\mathbf{X}$ is a symmetric matrix of dimensions $(|B||L|) \times (|B||L|)$. Let the $[(i,a),(j,b)]^{th}$ component of $\mathbf{X}$ be X_{ij}^{ab}. We now relax the constraint $\mathbf{X} = \mathbf{x}\mathbf{x}^T$ to obtain the following optimization problem:

$$\min \qquad Tr(\mathbf{W}\mathbf{X}) - \mathbf{u}^T\mathbf{x}, \tag{5.52}$$

$$\text{subject to} \qquad \mathbf{A}\mathbf{x} = \mathbf{1}, \tag{5.53}$$

$$\mathbf{X} \succcurlyeq \mathbf{x}\mathbf{x}^T, \tag{5.54}$$

$$\mathbf{x} \geqslant \mathbf{0}. \tag{5.55}$$

The constraint $\mathbf{X} \succcurlyeq \mathbf{x}\mathbf{x}^T$ means that $\mathbf{X} - \mathbf{x}\mathbf{x}^T$ is positive semidefinite. Problem (5.52) – (5.55) is a semidefinite relaxation of problem (5.48) – (5.51). The dual of problem (5.45) – (5.47) is as shown in (5.42) – (5.43). The formulation (5.42) – (5.43) can be reduced to a semidefinite programming problem, say D_{SDP}, using Shor's scheme shown below. This formulation is based on the formulation on Page 231 of Zheng et. al. (2012).

$$\mathbf{D_{SDP}}: \qquad \max \qquad \tau, \tag{5.56}$$

subject to
$$\begin{bmatrix} 0 & \frac{1}{2}\mathbf{u}^T \\ \frac{1}{2}\mathbf{u} & \mathbf{W} \end{bmatrix} - \tau \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \sum_{a \in L} \lambda^a \begin{bmatrix} -1 & A_a \\ A_a^T & \mathbf{0} \end{bmatrix} + \sum_{(i,a) \in BL} \psi_i^a \begin{bmatrix} 0 & \mathbf{b}^T \\ \mathbf{b} & \mathbf{0} \end{bmatrix} \succcurlyeq 0. \tag{5.57}$$

$$\tau \in \Re, \quad \lambda^a \in \Re \quad \forall a \in L, \quad \psi_i^a \geqslant 0 \quad \forall (i,a) \in BL, \tag{5.58}$$

where A_a is the a^{th} row of matrix $\mathbf{A}$ and is of dimensions $1 \times |BL|$, $\mathbf{b}$ is a column vector of 0s, except for the $(i,a)^{th}$ component which is 1, with dimensions $|BL| \times 1$, $\mathbf{0}$ is a square matrix of 0s whose dimensions are $|BL| \times |BL|$, and $\alpha \succcurlyeq 0$ means that matrix α is positive semidefinite. The dual of D_{SDP} is, in fact, problem (5.52) – (5.55). We refer readers to Page 61 in Boyd et. al. (1996) or Page 231 of Zheng et. al. (2012) for this. It is also true that, at optimality, the objective value of problem (5.52) – (5.55) equals the objective value of D_{SDP}. For this assertion, we refer readers to Page 61 of Boyd et. al. (1996).

Now, the optimal objective value of D_{SDP} is equal to the optimal objective value of problem (5.42) – (5.43). The optimal objective value of problem (5.42) – (5.43) is equal to the optimal objective value of problem QP by Theorem 5.3. From Theorem 5.2, the optimal solution and objective to QP is the same as that of QIP. Therefore, problem (5.52) – (5.55) solves QIP.

Finally, we know that problem (5.52) – (5.55) can be solved in polynomial time since it is a semidefinite programming problem (Boyd et. al. 1996). Therefore, QIP can be solved in polynomial time. □

The proof of Theorem 5.2 relies on two important results: (*i*) the optimal solution to QP is identical to that of QIP, and (*ii*) a semidefinite relaxation of QP, which can be solved in polynomial time, can solve problem QIP (Boyd et. al. 1996). The detailed proof is in the Appendix §**??**. The implication of Theorem 5.5 is that the problem of finding the social-welfare-maximizing allocation of items to bidders can be solved in polynomial time. Consequently, a direct mechanism with VCG allocation and payments can implement the social-welfare-maximizing allocation. We summarize this result in Corollary 5.2.

Corollary 5.2 *(Corollary to Theorem 5.5) Under PANE, the allocation and payments of a VCG auction can be computed in polynomial time.*

A VCG auction, by design, incentivizes bidders to report their valuations truthfully (Nisan et. al. 2007, Nisan and Ronen 2007). The auctioneer can use these truthful bids, i.e., $\mathbf{u}$ and $\mathbf{W}$, to solve QP in polynomial time using semidefinite programming, and obtain the optimal solution to QIP in the process. As a result, the computation of the VCG payments and allocation can be performed in polynomial time. Thus, a computationally efficient direct mechanism in the form of a VCG mechanism can be used to achieve a social-welfare-maximizing allocation under PANE.

5.5 The Subgradient Algorithm and the Subgradient Auction

In this section, we discuss the design of an iterative auction (an indirect mechanism) to achieve social welfare maximization. As mentioned in §5.1, iterative auctions may be preferred in practice because they do not require the bidders to reveal their private information completely. Also, iterative auctions are "decentralized" in the sense that the burden of computation of item allocation is delegated to the bidders: The bidders only have to compute their best responses to the current item prices in every round, while the auction design guides the movement of the prices in a manner that the final allocation achieves the auctioneer's goal (which, in our case, is maximizing social welfare). When the bidders' valuation function is PANE, the existence of ESAP shows that simple and anonymous prices are sufficient to guide the auction to a social-welfare-maximizing allocation. Thus, the iterative auction we discuss here only uses simple and anonymous prices.

We begin by describing a subgradient algorithm (Fisher 2004) as a solution technique to QIP since this algorithm, as we shall show, can be interpreted as an iterative auction. Problem QIP restated in matrix form is as follows:

$$\max_{\mathbf{x}} \mathbf{u}^T\mathbf{x} - \mathbf{x}^T\mathbf{W}\mathbf{x} \tag{5.59}$$

$$\text{subject to} \qquad \mathbf{A}\mathbf{x} = \mathbf{1}, \cdots (\mathbf{p}) \tag{5.60}$$

$$\mathbf{x} \in \{0,1\}^{|BL|\times 1}. \tag{5.61}$$

Relaxing constraint (5.60) using the multiplier $\mathbf{p}$, we have the following:

$$\mathcal{L}(\mathbf{x};\mathbf{p}) = \mathbf{u}^T\mathbf{x} - \mathbf{p}^T\mathbf{A}\mathbf{x} - \mathbf{x}^T\mathbf{W}\mathbf{x} + \mathbf{p}^T\mathbf{1}. \tag{5.62}$$

We now have the following optimization problem:

$$\min_{\mathbf{p}} \max_{\mathbf{x}} \mathcal{L}(\mathbf{x};\mathbf{p}), \tag{5.63}$$

$$\text{subject to} \qquad \mathbf{x} \in \{0,1\}^{|BL|\times 1}. \tag{5.64}$$

Theorem 5.3 shows that the subgradient algorithm solves the optimization problem (5.63) – (5.64) and will converge to a solution $\mathbf{x}^*$ that maximizes QIP. At iteration k, if the current value of the prices $\mathbf{p}^{(k)}$ and the current optimal solution to the inner optimization problem at $\mathbf{p}^{(k)}$ is $\mathbf{x}^{(k)*}$, $\mathbf{p}^{(k)}$ is adjusted by setting $\mathbf{p}^{(k+1)} = \mathbf{p}^{(k)} - \Delta^{(k)}(\mathbf{1} - \mathbf{A}\mathbf{x}^{(k)*})$. With appropriate step sizes (each component of $\Delta^{(k)}$ is the step size), the algorithm converges to the optimal solution. Each component of $\Delta^{(k)}$ is strictly positive. If a component of $\mathbf{1} - \mathbf{A}\mathbf{x}^{(k)*}$ is positive (a case of an item being demanded by no one), then the corresponding component of $\mathbf{p}^{(k+1)}$ is lower than that of $\mathbf{p}^{(k)}$, thus signifying a price reduction. Likewise, if a component of $\mathbf{1} - \mathbf{A}\mathbf{x}^{(k)*}$ is negative (a case of an item being demanded by more than one bidder), then the corresponding component of $\mathbf{p}^{(k+1)}$ is greater than that of $\mathbf{p}^{(k)}$, thus signifying a price increase. We refer readers to Fisher (2004) for details on the subgradient algorithm. This algorithm can be quite slow in convergence (Fisher 2004).

The subgradient algorithm described above can be interpreted as a guided-price auction[17]. This guided-price auction terminates with the social-welfare-maximizing allocation. The subgradient algorithm converges if it is allowed to use as many iterations as needed. For example, if each component of $\Delta^{(k)}$ was $\frac{1}{k}$, the algorithm would converge after a large number of iterations since $\lim_{n\to\infty} \sum_{k=1}^{k=n} \frac{1}{k} \to \infty$. Thus, this algorithm converges slowly if the prices are not updated appropriately.

The subgradient auction (or the guided-price auction). The auctioneer begins by setting prices p^a for all $a \in L$. Following this, she notes the interest the bidders have in the items at these prices. The expressions of interest of other bidders is made public by the auctioneer. Let $z_i^{a*} = 1$ if bidder i is interested in item a, and let $z_i^{a*} = 0$, if not, for all $a \in L$. A bidder i is interested in item a priced at p^a, $a \in L$ if

$$u_i^a - p^a - \sum_{\substack{(j,b)\in BL \\ j\neq i \\ b\neq a}} w_{ij}^{ab} z_j^{b*} > 0, \tag{5.65}$$

and is not interested in an item if not[18]. Following this, the auctioneer modifies the prices of each of the items and notes the bidders' interest again. This modification may be an increase or a decrease. She repeats this process until the following equation is satisfied,

$$\sum_{i\in B} z_i^{a*} = 1 \quad \forall\, a \in L,$$

i.e., only one bidder expresses interest in an item. At this point, the social-welfare maximizing allocation is reached. This auction format is truthful because no bidder would be willing to pay a price greater than his value for an item. This auction design is based on the subgradient algorithm. It terminates with a social-welfare-maximizing allocation. Figure 5.1 presents the steps of the subgradient auction. It starts with the auctioneer setting prices for all items. We discuss two illustrations of the subgradient auction in §5.6.

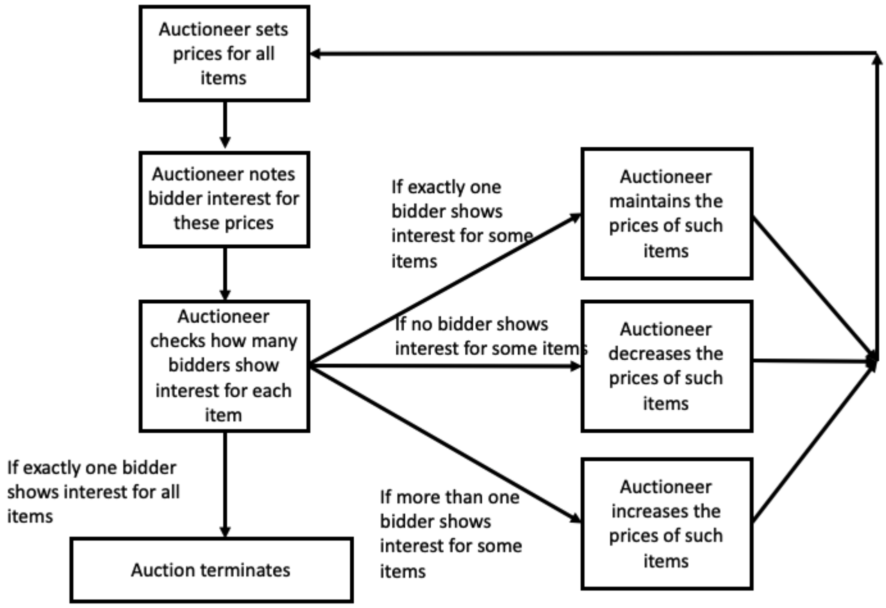


Figure 5.1: A schematic representation of the subgradient auction.

5.6 Numerical Experiments and Illustrations

In this section, we present demonstrations of the strong duality result of Theorem 5.3. We do this by implementing the subgradient algorithm to solve the minimax problem (5.63) – (5.64) and show that the optimal objective value of problem (5.63) – (5.64) is equal to the optimal objective value of QIP. We also present two illustrations of the subgradient auction. Both instances consider 3 items and 3 bidders, i.e., $L = \{a, b, c\}$ and $B = \{1, 2, 3\}$.

5.6.1 Demonstrating Strong Duality

We present our implementation of the subgradient algorithm to demonstrate strong duality in Algorithm 4 for four instances. The values of $\mathbf{u}$ and $\mathbf{W}$ are randomly-generated integers for all four instances. We run the computations using Python with CPLEX 20.1 (via the DOCPLEX package on Python) on a MacBook Air 2015 on the macOS Catalina operating system with a 1.6 GHz Dual-Core Intel Core i5 processor and a 4 GB 1600 MHz DDR3 RAM. In every iteration, we solve the binary quadratic optimization problem of Step 4 using CPLEX's inbuilt solver for binary quadratic programs[19]. The algorithm terminates after identifying an allocation $\mathbf{x}^*$ and maximum social welfare Z^* (both of which are defined in the algorithm).

Algorithm 4 The subgradient algorithm.

1. Initialize $\mathbf{p} = \langle M,\ a \in L \rangle$ where $M = \max_{(i,a)\in BL} u_i^a$

2. Initialize $k = 0,\ \Delta = 1,\ \mathbf{x}^* = \mathbf{0}$

while $\mathbf{1} - \mathbf{A}\mathbf{x}^* \neq \mathbf{0}$ **do**

3. $k = k + 1$

4. Find

$$Z^* = \max_{\mathbf{x}} \quad \sum_{(i,a)\in BL} \Big[u_i^a - p^a\Big] x_i^a - \sum_{(i,a)\in BL} \sum_{\substack{(j,b)\in BL \\ j\neq i \\ b\neq a}} w_{ij}^{ab} x_i^a x_j^b$$

$$\text{subject to} \quad x_i^a \in \{0,1\} \quad \forall\ (i,a) \in BL$$

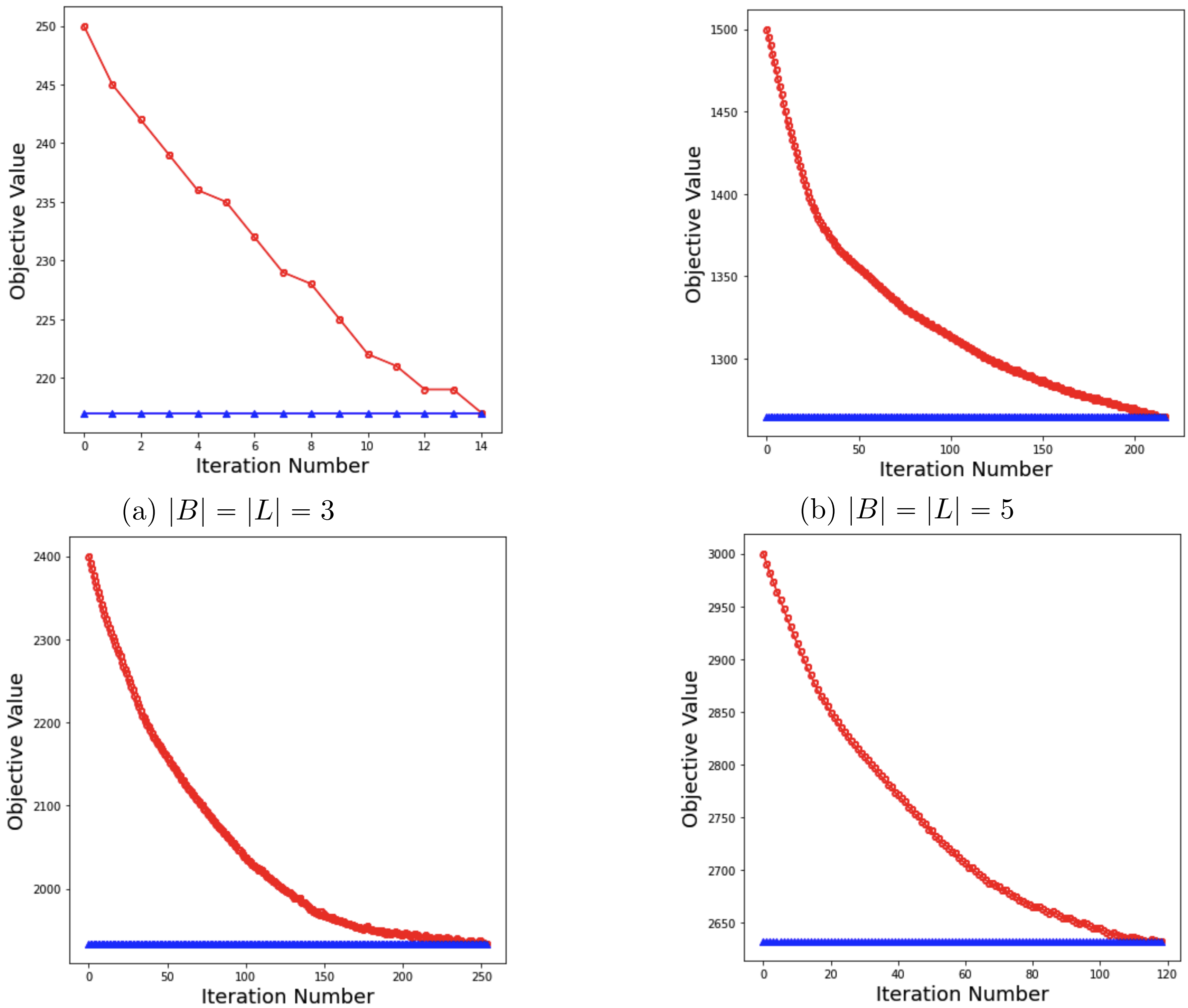
5. Set $Z^* = Z^* + \sum_{a\in L} p^a$

6. Set $p^a = \max\{0,\ p^a - \Delta(\mathbf{1} - \mathbf{A}\mathbf{x}^*)\}, \quad \forall\ a \in L$

end

7. Return $Z^*,\ \mathbf{x}^*$

Figure 5.2 depicts the progress of the subgradient algorithm over iterations for four problem instances. For all of the instances, $\Delta = 1$ for all iterations k. The curves in the figures (marked with an O; colored red) represent the value of the Lagrangean (5.63), i.e., Z^* in Algorithm 4, across iterations. The flat line segment at the bottom of the plots (marked with a Δ; colored blue) is the optimal objective value of QIP. The figures show how the Lagrangean objective (5.63) decreases across iterations, and intersects with the primal objective in the final iteration. Thus, these figures demonstrate strong duality as the value of the Lagrangean objective (5.63) converges to the optimal value of QP (and of QIP) in the final iteration.



(a) $|B| = |L| = 3$ (b) $|B| = |L| = 5$

(c) $|B| = |L| = 8$ (d) $|B| = |L| = 10$

Figure 5.2: The subgradient algorithm across iterations.

It is possible that setting $\Delta = 1$ for all k may cause the algorithm to cycle. It is, thus, important that an appropriate Δ be used for each iteration k. However, the underlying principle of the subgradient algorithm (and the suubgradient auction) continues to remain the same.

5.6.2 Illustrations of the Subgradient Auction

The problem instances generated for the two illustrations are as given in Tables 5.1 and 5.3 respectively. Tables 5.2 and 5.4 present the progression of the auction rounds for the two instances respectively. In Table 5.2 and Table 5.4, the first column is the 'Round Number'. It mentions the current round at which the subgradient auction runs. The second column is '$\mathbf{p}$' that presents the current prices of the items in a round. The third column is 'Undemanded'. The entries in this column are vectors where the a^{th}, $a \in L$ entry is defined as follows: If the a^{th} entry is 1, no bidder shows interest in the item. If the a^{th} entry is zero, exactly one bidder shows interest in item a. If the a^{th} entry is negative, more than one bidder shows interest in item a. The entries in the fourth column 'Demand' are vectors that depict the interest shown by the bidders for an item at prices $\mathbf{p}$: If the i^{th} bidder shows interest in item a, the $(i, a)^{th}$ entry in the vector is 1. Otherwise, it is zero.

u_i^a	a	b	c
1	8	8	5
2	5	6	7
3	8	9	7

(a) u_i^a values.

w_{ij}^{ab}	(1,a)	(1,b)	(1,c)	(2,a)	(2,b)	(2,c)	(3,a)	(3,b)	(3,c)
(1,a)	0	0	0	0	1	1	0	1	1
(1,b)	0	0	0	1	0	1	1	0	1
(1,c)	0	0	0	1	1	0	1	1	0
(2,a)	0	1	1	0	0	0	0	1	1
(2,b)	1	0	1	0	0	0	1	0	1
(2,c)	1	1	0	0	0	0	1	1	0
(3,a)	0	1	1	0	1	1	0	0	0
(3,b)	1	0	1	1	0	1	0	0	0
(3,c)	1	1	0	1	1	0	0	0	0

(b) w_{ij}^{ab} values.

Table 5.1: Illustration 1.

Round Number	$\mathbf{p}$	Undemanded	Demand
1	[10 10 10]	[1 1 1]	[0 0 0 0 0 0 0 0 0]
2	[9 9 9]	[1 1 1]	[0 0 0 0 0 0 0 0 0]
3	[8 8 8]	[0 0 1]	[0 0 0 0 0 0 1 1 0]
4	[8 8 7]	[0 0 0]	[0 0 0 0 0 0 1 1 1]

Table 5.2: The subgradient auction for illustration 1.

Illustration 1. In Table 5.2, the prices of all items start start at 10. At this price, no bidder shows any interest in the items. As a result, all items are undemanded. This is depicted by the vector [1 1 1]. Thus, the auctioneer reduces the price of each item by 1 unit. However, the undemanded item vector continues to be [1 1 1]. As a result, the auctioneer continues to lower the prices of each item by 1 unit. This time, there is a demand for the first two items a and b, while item c is undemanded. This is depicted by the vector [0 0 1]. As a result, the auctioneer reduces the price of item c by 1 unit. As a result, in round 4, all of the items are demanded at prices [8 8 7]. In this illustration, the allocation of the items was such that all of the items went to the same bidder. It can be verified easily from Table 5.1 that this is the social-welfare-maximizing allocation.

Illustration 2. The valuations and externalities for this illustration are as presented in Table 5.3, and the progression of the subgradient auction rounds are as presented in Table 5.4. The auctioneer sets the prices at 30 at the start of the auction, and modifies the prices depending on the interest for each of the items. From round 6 to round 7, the auctioneer lowers the prices of items a and c by 1 unit each since they remain undemanded at the prices [25 25 25]. From round 21 to round 22, the auctioneer increases the price of item a by 2 units, since there is an excess demand of 2 units for item a in round 21, as depicted by '-2' in the vector $[-2\ 1\ 1]$. At the same time, the auctioneer decreases the prices of items b and c by 1 unit each. This is because they are undemanded as of round 21. Likewise, from round 26 to round 27, the auctioneer increases the price of item b by 1 unit because it is overdemanded by 1 bidder. Finally, at prices [12 21 19], there is no undemanded item, and the assignments are such that items a and b go to bidder 1, and item c goes to bidder 2.

u_i^a	a	b	c
1	14	26	18
2	14	21	25
3	17	19	18

(a) u_i^a values.

w_{ij}^{ab}	(1,a)	(1,b)	(1,c)	(2,a)	(2,b)	(2,c)	(3,a)	(3,b)	(3,c)
(1,a)	0	0	0	0	1	1	0	2	1
(1,b)	0	0	0	2	0	1	2	0	2
(1,c)	0	0	0	2	1	0	1	1	0
(2,a)	0	1	1	0	0	0	0	1	1
(2,b)	1	0	2	0	0	0	2	0	2
(2,c)	1	1	0	0	0	0	2	1	0
(3,a)	0	2	2	0	1	2	0	0	0
(3,b)	1	0	2	1	0	1	0	0	0
(3,c)	1	1	0	1	1	0	0	0	0

(b) w_{ij}^{ab} values.

Table 5.3: Illustration 2.

We point out that changing prices in multiples of 1 unit between rounds may lead to cycling. In general, this can be circumvented by changing the item prices as a multiple of $\frac{1}{k}$, where k is the current round number. We point out that, given a price-update structure, it need not be the case that the final item prices are ESAP, although it is guaranteed that the final allocation is the social-welfare-maximizing

Round Number	$\mathbf{p}$	Undemanded	Demand
1	[30 30 30]	[1 1 1]	[0 0 0 0 0 0 0 0 0]
2	[29 29 29]	[1 1 1]	[0 0 0 0 0 0 0 0 0]
$\vdots$	$\vdots$	$\vdots$	$\vdots$
5	[26 26 26]	[1 1 1]	[0 0 0 0 0 0 1 1 0]
6	[25 25 25]	[1 0 1]	[0 1 0 0 0 0 0 0 0]
7	[24 25 24]	[1 0 1]	[0 1 0 0 0 0 0 0 0]
$\vdots$	$\vdots$	$\vdots$	$\vdots$
9	[22 24 23]	[1 0 0]	[0 1 0 0 0 1 0 0 0]
$\vdots$	$\vdots$	$\vdots$	$\vdots$
21	[13 22 21]	[-2 1 1]	[1 0 0 1 0 0 1 0 0]
22	[15 21 20]	[1 0 0]	[0 1 0 0 0 1 0 0 0]
$\vdots$	$\vdots$	$\vdots$	$\vdots$
26	[14 20 19]	[1 -1 0]	[0 1 0 0 1 1 0 0 0]
27	[13 21 19]	[1 0 0]	[0 1 0 0 0 1 0 0 0]
28	[12 21 19]	[0 0 0]	[1 1 0 0 0 1 0 0 0]

Table 5.4: The subgradient auction for illustration 2.

allocation. This is because multiple item prices can give rise to the same allocation. This follows from the fact that dual formulation of (5.5) – (5.8) has multiple optimal solutions.

5.7 Conclusion and Future Research

We examine the auctioneer's problem of designing auction mechanisms (direct and indirect) that terminate in a social-welfare maximizing allocation for settings with negative externalities. Specifically, we consider a class of valuation functions we call the Pairwise Additive Negative Value Externalities (PANE). We formulate the social-welfare maximization problem as a binary quadratic programming problem with linear constraints. We show that the objective function is generally neither convex nor concave. We also identify some important properties of this class of optimization problems: (*i*) The optimal solution to the social-welfare maximization problem is binary even after the binary constraints are relaxed, (*ii*) strong duality holds true for this class of optimization problems, and (*iii*) this class of optimization problems can be solved in polynomial time. Results (*i*) and (*ii*) are important for showing the existence of simple and anonymous prices that support a social-welfare-maximizing allocation of items, thus aiding in the development of practically-implementable auction designs that terminate with a social-welfare-maximizing allocation. Result (*iii*) shows that a VCG auction can be conducted in polynomial time, i.e., the allocation and bidder payments can be computed in polynomial time. We note that the strong duality result of Theorem 5.3 is driven primarily by the semi-assignment constraints (5.22).

We show how the subgradient algorithm (Fisher 2004) can be used to solve the social welfare maximization problem. For this, we make use of the fact that the relaxation of the social-welfare maximization problem possesses strong duality, and that this problem solves the social welfare maximization problem. We interpret the subgradient algorithm as a guided-price auction, and present illustrations explaining the auction process.

We also present numerical demonstrations of the strong duality property by implementing the subgradient algorithm. A key step of the subgradient algorithm is the updating of the item prices. It may be the case that the algorithm cycles under some choices of price-updates although it is possible to circumvent this issue by choosing an appropriate price-update structure. However, we do not look into the question of recommending a price-update structure, as we focus primarily on the design of the auction. The operating principle of the subgradient algorithm is the same regardless of the price-update structure used, and we leave the question of finding an appropriate price-update structure to future research. We also conjecture that the subgradient algorithm (and the subgradient auction) would terminate with a social-welfare-maximizing allocation for any function $H(\mathbf{x}) = \sum_{i \in B} v_i(\mathbf{x})$ that satisfies the conditions in Theorem 5.1. However, we leave a full analysis of this conjecture to future research.

As contribution to literature on auction designs with allocative externalities, we present a notion of equilibrium that is applicable in settings where allocative externalities are present. In addition, we present conditions on the bidders' valuation functions for which simple and anonymous prices are sufficient to achieve a social-welfare-maximizing allocation.

As part of future research, one can consider examining other kinds of valuation functions that account for allocative externalities and analyze structural properties of the social-welfare-maximization problem. Based on insights from such analyses, one can consider appropriate auction designs. In particular, if for some classes of valuation functions with allocative externalities, simple and anonymous prices cannot result in a social-welfare-maximizing allocation, one can study the types of higher order pricing schemes that support a social-welfare-maximizing allocation. Our work provides the starting point for future research in these lines.

Notes

[11]We use the terms slot and item interchangeably throughout the chapter.

[12]We use the terms advertisers, bidders, and agents interchangeably throughout the text.

[13]We use the pronoun 'she' to refer to the auctioneer, and the pronoun 'he' to refer to the bidders.

[14]This terminology was earlier used in Candogan et. al. (2015). However, I formally define these terms later for clarity.

[15]GSP stands for Generalized Second Price

[16]See Bikhchandani et. al. (2002) for discussions on complex pricing structures.

[17]We refer to this auction format as the guided-price auction or subgradient auction interchangably throughout this chapter.

[18]Alternatively, we say that bidder i *demands* item a if bidder i is interested in item a, and that he *does not demand* item a if he is not interested in it.

[19]https://www.ibm.com/docs/en/icos/20.1.0?topic=parameters-optimality-target

Chapter 6

Conclusions and Future Work

The theme of this thesis is on designing mechanisms for multi-item sales in contexts that have not been studied in much detail in literature. It examines two kinds of questions of interest to an auctioneer in relation to multi-item sales: (*i*) revenue enhancement and (*ii*) identifying a social-welfare-maximizing allocation.

Chapters 3 and 4 examine the question of revenue enhancement in the context of multi-item sales. Both chapters study supply control as a revenue-enhancement lever. Chapters 3 looks at bundling as a lever, while Chapter 4 looks at identifying a subset of items to put on offer as a lever. Chapter 5 looks at the problem of ideitifying a social-welfare-maxmizing allocation in settings with allocative externalities.

The key contributions of Chapter 3 are about (*i*) showing the existence of a truthful mechanism where item-level bids are collected *before* the items are bundled, and (*ii*) presenting a formulation for computing such item bundlings that can be solved using standard solvers for integer programming. The key contribution of Chapter 4 is to show how an optimal offer set can be computed in polynomial time if the state space considered is polynomially bounded. The key contributions of Chapter 5 are about (*i*) identifying a class of binary quadratic optimization problems whose binary relaxation yields binary optimal solutions, (*ii*) identifying a class of nonconvex quadratic optimization problems with strong duality, and (*iii*) identifying a class of nonconvex quadratic optimization problems that can be solved in polynomial time.

Chapter 3 discusses a revenue-enhancement mechanism where bidders' bids are collected before an allocation rule is decided. In this regard, future research can examine mechanism design problems for revenue-enhancement where bids are collected *before* an allocation rule is decided. Such a means of revenue-enhancement has not been explored in literature, and can potentially lead to better-capturing of the bidders' surpluses, leading to higher revenues for the auctioneer.

Chapter 4 discusses revenue-enhancement where the item set on offer could be chosen to minimize the costs of participation uncertainty. A takeaway from this chapter is that the bidders' participation decisions are affected by the rules of the sale mechanism. This requires designing approaches to compute an offer set as a function of the rules of the sale mechanism. The question of optimizing offer sets can be explored for other commonly-used sale mechanisms, particularly those for combinatorial auctions.

Chapter 5 discusses the conditions for the existence of simple and anonymous equilibria for general valuation functions, and shows that they exist under PANE. Future research could explore other relevant forms of valuation functions, and examine whether equilibrium allocations are supported by simple and anonymous pricing schemes. Alternatively, future research could explore more complex pricing schemes to achieve social-welfare-maximizing allocations under other valuation functions.

Thus, my thesis has explored important questions around sale mechanisms for multiple items in contexts that have not been explored in literature, and has suggested a promising line of future work.

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